

# TOPOLOGICAL RADICALS, V. FROM ALGEBRA TO SPECTRAL THEORY

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*To the memory of Bill Arveson, a great mathematician and a great person*

**ABSTRACT.** We introduce and study procedures and constructions of the theory of general topological radicals that are related to the spectral theory — the centralization, primitivity and socle procedures, the scattered radical, the radicals related to the continuity of the usual, joint and tensor radii. Among other applications we find some sufficient conditions of continuity of the spectrum and spectral radii of different types, and in particular prove that in a GCR  $C^*$ -algebra the joint spectral radius is continuous on precompact sets and coincides with the Berger-Wang radius.

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## 1. INTRODUCTION AND PRELIMINARIES

**1.1. Introduction.** The general theory of radicals can be considered as a global structure theory of algebras which is aimed at the study of ideals and quotients simultaneously defined for a large class of algebras and related to some general properties of algebras.

Briefly speaking, a map  $P$  defined on some class  $\mathfrak{U}$  of algebras and sending each algebra  $A \in \mathfrak{U}$  to its ideal  $P(A)$  is called a radical if it satisfies some natural conditions of covariance and stability (see Section 2.1). The most popular example of a radical is the *Jacobson radical*  $\text{rad}$  defined on the class of all algebras; its restriction to the class of all Banach algebras is denoted by  $\text{Rad}$ . The importance of this map for the theory of Banach algebras cannot be overestimated.

The first significant achievements of the theory of radicals were related to the nilpotency and some close properties, and obtained by Baer, Levitzki, Köthe, Amitsur, Kurosh and others prominent algebraists. The functional analytic counterpart of this theory was initiated by Peter Dixon in his paper [D3], which contained basic definitions and presented first applications. Since then the theory was developed and applied to different problems of operator theory and Banach algebras in [ST2, ST5, ST6, ST7, KST]. The problems that turned out to be "solvable in radicals" had their origin in the theory of invariant subspaces, irreducible representations, semigroups of operators, tensor products, linear operator equations, joint spectral radius, Banach Lie algebras and other topics. They lead to introducing and study of corresponding radicals – the (*closed-*) *hyponfinite radical*  $\mathcal{R}_{\text{hf}}$ , the *hypocompact radical*  $\mathcal{R}_{\text{hc}}$ , the *tensor radical*  $\mathcal{R}_t$ , the *compactly quasinilpotent radical*  $\mathcal{R}_{\text{cq}}$  and others (see Section 2 for definitions and discussion of these and other examples). It can be said that the essence of the radical approach to a problem (a property, a phenomenon) is to single out an ideal that accumulates elements related to this problem, and study the dependence of this ideal on the algebra. The construction of the corresponding radical is a typical result in the theory, and this is reasonable because all "well behaved" radicals in their time find applications.

For functional analysts, the radicals on Banach algebras are most interesting. However, to have a more flexible technique, one has to consider topological radicals

defined on non-complete normed algebras and also algebraic radicals defined on algebras without topology. For instance, the study of non-closed ideals of a Banach algebra ideals which are complete in a larger norm (we call them Banach ideals) turns out to be very useful. This is a reason to consider radicals in the wider context. So we outline three levels of consideration of radicals: algebraic, normed and Banach. The relations between these three theories are quite complicated. For example, the restriction of  $\text{rad}$  to the class of *all* normed algebras cannot be considered as a topological radical, because  $\text{rad}(A)$  can be non-closed in  $A$  if  $A$  is not complete.

In this paper our main objects are spectral characteristics — spectrum, primitive ideals, spectral radii of different nature, socle, nilpotency, spectral continuity etc — and we construct special radicals and general procedures of the theory of radicals aiming at applications to the spectral theory in its algebraic and functional analytic aspects. The interplay between algebraic and functional analytic sides of the spectral theory forces us to devote a large part of the paper to understanding the links between algebraic and topological theories of radicals.

All main objects and constructions are related, to a greater or lesser extent, to the *scattered radical*  $\mathcal{R}_s$ . This radical associates with any Banach algebra the largest ideal whose elements have at most countable spectrum. From the viewpoint of the hierarchy of radicals the formulas

$$\mathcal{P}_\beta = \mathcal{R}_{\text{hf}} \wedge \text{Rad}, \quad (1.1)$$

$$\mathcal{R}_s = \mathcal{R}_{\text{hf}} \vee \text{Rad} \quad (1.2)$$

exactly determine the place of the signed radicals in Banach algebras: (1.1) and (1.2) mean that the *closed-Baer radical*  $\mathcal{P}_\beta$  is the infimum of  $\mathcal{R}_{\text{hf}}$  and  $\text{Rad}$ , and  $\mathcal{R}_s$  is the supremum of the same pair of radicals. All these radicals are topological, but  $\mathcal{R}_s$  and  $\text{Rad}$  are the restrictions of some algebraic radicals to Banach algebras while  $\mathcal{P}_\beta$  and  $\mathcal{R}_{\text{hf}}$  have algebraic analogs.

The paper is organized as follows. In the first section we gather the necessary preliminary information. The second section contains the basic definitions of the radical theory and the discussion of additional properties of radical-like maps. We present here also the basic examples of topological radicals. We introduce also radicals defined on the class of  $C^*$ -algebras.

We call by a *procedure* any rule that transforms ideal maps. The simplest one which we use by default is the *restriction* of radicals to a subclass of algebras. Section 3 is devoted to the study of several important procedures. The first two of them (the *convolution procedure*  $P \mapsto P^*$  and *superposition procedure*  $P \mapsto P^\circ$ ) were introduced by Baer in the purely algebraic context, and by Dixon in the topological one. They produce a radical from an ideal map that lacks some stability property — an under radical and an over radical; we complement Dixon's results for the case of normed algebras. The third one (the *closure*) is very simple — from a map  $A \mapsto P(A)$  with non-necessarily closed ideals  $P(A)$  it makes the map  $A \mapsto \overline{P(A)}$  which can be a topological radical or be transformed into a topological radical by means of the convolution procedure. Thus by means of the closure one can obtain a topological radical from an algebraic one. The fourth procedure is the *regularization*  $P \mapsto P^r$ ; it allows one to extend a radical from the class of all Banach algebras to a class of normed algebras by taking the completion.

In Section 4 we study some operations (multiplace procedures), that produce new radicals or radical-like maps from given families of radicals: supremum, infimum and two-place procedures — the convolution operation  $*$  and superposition operation  $\circ$ . One of our aims here is to show that they all are closely related. For instance, the convolution and superposition procedures are the results of transfinite

applications of the convolution and superposition operations. Supremum is reached by the convolution procedure; respectively, infimum is reached by the superposition procedure. We also study conditions which imply that a class of algebras is radical or semisimple.

We consider the problem of heredity of a radical that is obtained by closure and convolution procedures. A radical  $P$  is called *hereditary* if  $P(J) = J \cap P(A)$  for any appropriate ideal  $J$  of an algebra. This property is very convenient for the study and use of a radical. Since many (almost all) most important radicals are constructed by means of the closure and convolution, the heredity problem is one of the main inner problems of the theory. We give a criterion of heredity for the resulting radical which works for almost all known examples. In particular this approach allows us to answer in affirmative the question of Dixon [D3] about the heredity of the closed-Baer radical  $\mathcal{P}_\beta$ . This radical is the smallest one among topological radicals on Banach algebras for which all algebras with trivial multiplication are radical.

In Section 5 we deal with the radical approach to the property of commutativity modulo a radical. Namely, starting with a radical  $P$  we define an under radical  $P^a$ , such that  $P^a$ -radical algebras are precisely those algebras that are commutative modulo  $P$ :

$$[A, A] \subset P(A).$$

We find the conditions on  $P$  under which  $P^a$  is a radical, and check that they are fulfilled for our main examples. For instance, this is true for all algebraic hereditary radicals and their topological analogs obtained by the closure and convolution procedures. In particular,  $\text{rad}^a$ ,  $\mathcal{R}_s^a$ ,  $\mathcal{P}_\beta^a$ ,  $\mathcal{R}_{\text{hf}}^a$ ,  $\mathcal{R}_{\text{cq}}^a$  and  $\mathcal{R}_t^a$  are radicals. While the first four radicals satisfy our criterion, the last two need a separate consideration (the result for  $\mathcal{R}_{\text{cq}}^a$  was proved in [ST7]). This underlines the advantage of the joint consideration of algebraic and topological radicals. As the commutativity modulo  $\text{rad}$ ,  $\mathcal{R}_{\text{cq}}$  and  $\mathcal{R}_t$  is often used in the theory of spectral radii of different types, some important applications to spectral theory are indicated at the end of the section and also in Section 9. In particular, we consider the sufficient conditions for a Banach algebra to be Engel.

The subject of Section 6 is the very popular in the theory of algebras and Banach algebras notion of the socle of an algebra. We consider the ideal map  $\text{soc} : A \longrightarrow \text{socle}(A)$  and define a new procedure that transforms each radical  $P$  into the convolution  $\text{soc} * P$  (“the socle modulo  $P$ ”). By means of this procedure we establish some relations between, for example, hypofinite radical and Baer radical. Main applications of this construction will be given in Section 8 where we introduce the scattered radical.

Our object in Section 7 is more spectral: we study procedures related to the space  $\text{Prim}(A)$  of all primitive ideals of an algebra  $A$ . All of them begin with a choice of a subset  $\Omega(A) \subset \text{Prim}(A)$ , but such a choice (a *primitive map*) must be done simultaneously for all algebras  $A$  in the given class, and subjected to some natural restrictions. For example one can choose for  $\Omega(A)$  all the space  $\text{Prim}(A)$ , or  $\emptyset$ , or the set of primitive ideals of finite codimension. If a primitive map is fixed then the procedures we deal with are

- The  $\Omega$ -hull-kernel closure  $P^{\text{kh}\Omega}$  of a preradical  $P$  which sends  $A$  to the intersection of those ideals  $I \in \Omega(A)$  that contain  $P(A)$ ;
- The  $\Omega$ -primitivity extension  $P^{p\Omega}$  of  $P$  that sends  $A$  to the ideal of all elements  $a \in A$  such that  $a/I \in P(A/I)$ , for all  $I \in \Omega(A)$ .

Our main aims here are to study the interrelations between these procedures and to find the conditions on  $P$  under which the produced maps have sufficiently convenient

properties. Note that the primitivity extension of the scattered radical take a central place in applications to the spectral continuity.

Section 8 which can be considered central in this paper, is devoted to the theory of the scattered radical  $\mathcal{R}_s$ . Almost all constructions of the previous sections find a reflection in or application to the theory of  $\mathcal{R}_s$ . We introduce and study  $\mathcal{R}_s$  in its algebraic and Banach algebraic versions. The algebraic version of  $\mathcal{R}_s$  is defined via the socle and convolution procedure: it equals  $(\text{soc} * \text{rad})^*$ . We show, using the connection between both versions, that  $\mathcal{R}_s$  is a hereditary topological radical satisfying some additional flexibility conditions which allow one to use its centralization and primitivity extensions.

Another important fact is that the structure space of an  $\mathcal{R}_s$ -radical algebra is dispersed and that (the classes of the equivalence of) strictly irreducible representations of such algebras are uniquely determined by their kernels (= primitive ideals). We show that in hereditarily semisimple Banach algebras (in particular, in  $C^*$ -algebras)  $\mathcal{R}_s$  coincides with the hypocompact radical and that scattered  $C^*$ -algebras can be characterized by many other equivalent conditions.

The subject of Section 9 is the continuity of spectral characteristics of an element or a family of elements of a Banach algebra. We show that the map  $a \mapsto \sigma(a)$  is continuous at points  $a \in \mathcal{R}_s^p(A)$  where  $\mathcal{R}_s^p$  is the primitivity extension of the scattered radical, and that this property is not fulfilled for elements of  $\mathcal{R}_s^{p*}(A)$  and of  $\mathcal{R}_s^a$ . Among other results we obtain that if  $A$  is a  $C^*$ -algebra then  $\mathcal{R}_s^{p*}(A)$  is the largest GCR-ideal of  $A$  (so the radical  $\mathcal{R}_s^{p*}$  extends the GCR-radical from  $C^*$ -algebras to Banach algebras).

We construct topological radicals  $\mathcal{R}_{\vec{\rho}}$ ,  $\mathcal{R}_{\vec{\rho}_j}$  and  $\mathcal{R}_{\vec{\rho}_t}$  that have the properties that elements, compact subsets and summable families in  $\mathcal{R}_{\vec{\rho}}(A)$ ,  $\mathcal{R}_{\vec{\rho}_j}(A)$  and  $\mathcal{R}_{\vec{\rho}_t}(A)$ , respectively, are the continuity points for spectral radius, joint spectral radius and tensor spectral radius, respectively. It is shown that  $\mathcal{R}_{\vec{\rho}} \geq \mathcal{R}_s^{p*}$ . Thus the spectral radius, unlike the spectrum, is continuous at all elements of  $\mathcal{R}_s^{p*}(A)$ . In particular,  $\rho$  is continuous on any GCR  $C^*$ -algebra.

In Section 10 we apply some of the listed results to the problem of recovering the spectral characteristics of an element or a family of elements of an algebra  $A$  from the corresponding information about its images in some quotients  $A/I$ , where  $I$  belong to a fixed family  $\mathcal{F}$  of ideals in  $A$  with trivial intersection. Our main interest is in the joint spectral radius  $\rho(M)$  of a precompact set  $M \subset A$  — a characteristic that attracts much interest not only in operator theory but in such branches of mathematics as the topological dynamics and fractal theory. We show that

$$\rho(M) = \sup_{I \in \mathcal{F}} \rho(M/I) \quad (1.3)$$

if  $\mathcal{F}$  is finite, and extend this equality to the case of infinite family  $\mathcal{F}$  for algebras that have non-zero compact elements. An especially interesting case is  $\mathcal{F} = \text{Prim}(A)$ . We relate (1.3) to the equality  $\rho(M) = r(M)$ , where  $r(M)$  is a spectral characteristic introduced by Berger and Wang [BW]; in the algebras where this equality holds for all precompact subsets (the *Berger-Wang algebras*) the analysis of many spectral problem becomes much easier.

We show that in the  $\mathcal{R}_s^{p*}$ -radical Berger-Wang algebras the joint spectral radius is continuous and find some criteria for (1.3) to hold. Then we study the case of  $C^*$ -algebras and show that each GCR-algebra is a Berger-Wang algebra and therefore (if one takes into account that GCR-algebras are  $\mathcal{R}_s^{p*}$ -radical) the joint spectral radius is continuous on GCR-algebras.

It is natural that simultaneous consideration of various kinds of radicals — the algebraic ones, the radicals on Banach algebras, normed algebras,  $Q$ -algebras,  $C^*$ -algebras and so on — can be difficult for the first acquaintance with the topic.

However, apart from the reasons above, it gives the possibility not to repeat similar arguments many times. Anyway for the first reading it seems to be reasonable to restrict oneself to a concrete type of radicals and consider only Banach or  $C^*$ -algebras. We can say that we did our best to simplify the text and, in particular, following the advice of Laurence Sterne, generously lighted up the dark places by stars.

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## 1.2. Preliminaries.

1.2.1. *Spaces.* Let  $X, U$  be linear spaces, and let  $Y$  and  $Z$  be subspaces of  $X$ . If  $Z \subset Y$  then  $(X/Z)/(Y/Z) \cong X/Y$ . If  $(Y_\alpha)_{\alpha \in \Lambda}$  is a family of subspaces of  $X$  and  $Z \subset \bigcap_{\alpha \in \Lambda} Y_\alpha$  then

$$(\bigcap_{\alpha \in \Lambda} Y_\alpha)/Z = \bigcap_{\alpha \in \Lambda} (Y_\alpha/Z). \quad (1.4)$$

If  $X$  is normed, let  $\overline{Y}$ , or more exactly  $\overline{Y}^{(X)}$ , denote the closure of  $Y$  in  $X$ , and  $\widehat{X}$  denote the completion of  $X$ . If  $Z \subset Y$  is a closed subspace of  $Y$  then  $Y/Z \cong q(Y)$ , where  $q : X \rightarrow X/\overline{Z}^{(X)}$  is the standard quotient map  $x \mapsto$  the coset of  $x$ .

Let  $f : X \rightarrow U$  be a linear map. If  $M \subset U$  then

$$f^{-1}(M) := \{x \in X : f(x) \in M\}.$$

Clearly  $N \subset f^{-1}(f(N))$  for every  $N \subset X$ ;  $N = f^{-1}(f(N)) \Leftrightarrow N = f^{-1}(M)$  for some  $M \subset U$ ; in particular this holds if  $f$  is invertible. If  $X, U$  are normed, and  $f$  is open and continuous, then

$$f^{-1}(\overline{M}) = \overline{f^{-1}(M)}. \quad (1.5)$$

If  $X$  is a linear space then  $L(X)$  is the algebra of all linear operators on  $X$ ,  $\mathcal{F}(X)$  the ideal of finite rank operators. If  $X$  is normed then  $\mathcal{B}(X)$  is the algebra of all bounded operators,  $\mathcal{K}(X)$  is the ideal of compact operators.

1.2.2. *Algebras.* Let  $A$  be an associative complex algebra; then  $A^1$  equals  $A$  if  $A$  is unital, and is the algebra obtained from  $A$  by adjoining the identity element otherwise. In what follows, an *ideal* of an algebra means a two-sided ideal. If  $I$  is an ideal of  $A$  then  $q_I$  denotes the standard quotient map  $A \rightarrow A/I$  by default; we also write  $a/I$  instead of  $q_I(a)$  for any  $a \in A$ . Respectively, if  $M = (a_\alpha)_\Lambda$  is a family in  $A$  then  $M/I$  denotes the family  $(a_\alpha/I)_\Lambda$  in  $A/I$ . Define operators  $L_a, R_a$  and  $W_a$  on  $A$  by

$$L_a x = ax, R_a x = xa \text{ and } W_a = L_a R_a;$$

again  $L_M$  is a family  $(L_{a_\alpha})_\Lambda$  of operators. Such rules act for sets by default.

1.2.3. *Representations.* A representation  $\pi$  of an algebra  $A$  on a linear space  $X$  is called *strictly irreducible* if  $\pi(A)\xi := \{\pi(a)\xi : a \in A\} = X$ , for every  $\xi \neq 0$ . Representations  $\pi$  and  $\tau$ , acting on the spaces  $X$  and  $Y$ , respectively, are *equivalent* (write  $\pi \sim \tau$ ) if there is a linear bijective operator  $T : X \rightarrow Y$  satisfying the condition  $T\pi(a) = \tau(a)T$  for all  $a \in A$ . The direct sum  $\bigoplus_{i=1}^n \pi_i$  of representations  $\pi_1, \dots, \pi_n$ , acting on the spaces  $X_1, \dots, X_n$ , is the representation  $\pi$  on  $X = \bigoplus_{i=1}^n X_i$  defined by the formula

$$\pi(a)(\bigoplus_{i=1}^n \xi_i) = \bigoplus_{i=1}^n \pi_i(a)\xi_i.$$

It is well known (see for example [Ln, Chapter 17]) that if strictly irreducible representations  $\pi_1, \dots, \pi_n$  are pairwise nonequivalent and  $0 \neq \eta_i \in X_i$ ,  $1 \leq i \leq n$ , then the vector  $\eta = \bigoplus_{i=1}^n \eta_i$  is cyclic for the representation  $\pi$ :  $\pi(A)\eta = X$ .

**Lemma 1.1.** *Let  $\{\pi_i : 1 \leq i \leq n\}$  be mutually inequivalent strictly irreducible representations. Then  $\sum_{i=1}^n \text{rank}(\pi_i(a)) \leq \dim aAa$  for any  $a \in A$ .*

*Proof.* Fix  $a \in A$ . We may suppose that  $\dim aAa < \infty$  and that  $\pi_i(a) \neq 0$  for all  $i$ .

Let  $\pi = \oplus_{i=1}^n \pi_i$  and  $X = \oplus_{i=1}^n X_i$ . Clearly

$$\dim \pi(aAa)\xi \leq \dim aAa$$

for each  $\xi \in X$ . Let us choose vectors  $\xi_i \in X_i$  such that  $\eta_i := \pi_i(a)\xi_i \neq 0$ ,  $1 \leq i \leq n$ . Let  $\xi = \oplus_{i=1}^n \xi_i$ ; then the vector  $\eta = \pi(a)\xi = \oplus_{i=1}^n \pi_i(a)\xi_i$  is cyclic for  $\pi(A)$ . Therefore  $\pi(aAa)\xi = \pi(a)\pi(A)\eta = \pi(a)X = \oplus_{i=1}^n \pi_i(a)X_i$  whence

$$\dim \pi(aAa)\xi = \sum \dim \pi_i(a)X_i = \sum_{i=1}^n \text{rank}(\pi_i(a)).$$

□

Let  $\text{Irr}(A)$  be the set of all strictly irreducible representations of  $A$ ,  $\text{Prim}(A)$  the set of all primitive ideals, i.e., the kernels of all  $\pi \in \text{Irr}(A)$ . For any subset  $E$  of  $A$ , let  $\text{h}(E; A)$  be the set of all  $I \in \text{Prim}(A)$  with  $E \subset I$  and  $\text{kh}(E; A) = \cap_{I \in \text{h}(E; A)} I$ .

If  $\pi \in \text{Irr}(A)$  then let  $X_\pi$  be the space on which  $\pi$  acts (the representation space). It is well known that if  $A$  is not unital then

$$\text{Irr}(A^1) = \{\pi_{\text{triv}} : A^1 \longrightarrow A^1/A \cong \mathbb{C}\} \cup \{\pi^1 : \pi \in \text{Irr}(A)\}, \quad (1.6)$$

$$\text{Prim}(A^1) = \{A\} \cup \text{Prim}(A) \text{ and } \text{rad}(A^1) = \text{rad}(A) \quad (1.7)$$

where every  $\pi \in \text{Irr}(A)$  extends to  $\pi^1 \in \text{Irr}(A^1)$  by setting  $\pi^1(1_{A^1}) = 1_{X_\pi}$ .

If  $J$  is a proper ideal of  $A$  then

$$\text{Irr}(J) = \{\pi|_J : \pi \in \text{Irr}(A), \pi(J) \neq 0\}, \quad (1.8)$$

$$\text{Prim}(J) = \{I \cap J : I \in \text{Prim}(A), I \cap J \neq J\}. \quad (1.9)$$

Every  $\pi \in \text{Irr}(J)$  can be uniquely extended to a representation  $\tilde{\pi} \in \text{Irr}(A)$ . To construct the extension it suffices to choose a non-zero vector  $\zeta \in X_\pi$  and, for each  $a \in A$ , to set  $\tilde{\pi}(a)\xi = \pi(ab)\zeta$  for every  $b \in J$  such that  $\xi = \pi(b)\zeta$ .

Every  $\pi \in \text{Irr}(A/J)$  induces  $\pi_A \in \text{Irr}(A)$  by setting  $\pi_A(a) = \pi(a/J)$  for every  $a \in A$ . Every  $\pi \in \text{Irr}(A)$  with  $\pi(J) = 0$  induces  $\pi_{q_J} \in \text{Irr}(A/J)$  by setting  $\pi_{q_J}(q_J(a)) = \pi(a)$  for every  $a \in A$ . Hence

$$\text{Prim}(A/J) = \{I/J : I \in \text{Prim}(A), J \subset I\}. \quad (1.10)$$

Note that  $A/J$  can be unital for an non-unital algebra  $A$ .

If  $f : A \longrightarrow B$  is a homomorphism onto then  $\pi \in \text{Irr}(B)$  induces  $\pi^f \in \text{Irr}(A)$  by setting  $\pi^f(a) = \pi(f(a))$  for every  $a \in A$ . So

$$\{f^{-1}(I) : I \in \text{Prim}(B)\} \subset \text{Prim}(A). \quad (1.11)$$

Let  $A$  be a normed algebra, let  $\text{Irr}_n(A)$  (respectively,  $\text{Irr}_b(A)$ ) be the set of all continuous strictly irreducible representations of  $A$  by bounded operators on a normed (respectively, Banach) space, and let  $\text{Prim}_n(A) = \{\ker \pi : \pi \in \text{Irr}_n(A)\}$  and  $\text{Prim}_b(A) = \{\ker \pi : \pi \in \text{Irr}_b(A)\}$ .

It should be noted that for a  $C^*$ -algebra, a primitive ideal is defined as the kernel of an irreducible  $*$ -representation on a Hilbert space. The fact that this definition is equivalent to the general one, is proved in [Dm, Corollary 2.9.6].

**1.2.4.  $Q$ -algebras.** A normed algebra  $A$  is called a  $Q$ -algebra if the set of all invertible elements of  $A^1$  is open. A normed algebra  $A$  is a  $Q$ -algebra  $\Leftrightarrow \sum_{n=1}^{\infty} a^n$  converges for any  $a \in A$  with  $\|a\| < 1 \Leftrightarrow$  every strictly irreducible representation of  $A$  is equivalent to a continuous representation by bounded operators on a normed space [ST5, Theorem 2.1]. A normed algebra  $A$  is called a  $Q_b$ -algebra if every strictly irreducible representation of  $A$  is equivalent to a continuous representation by bounded operators on a Banach space.

Let  $A$  be an algebra and  $a \in A$ . The *spectrum*  $\sigma(a)$ , or more exactly  $\sigma_A(a)$ , is the set of all  $\lambda \in \mathbb{C}$  such that  $a - \lambda$  is not invertible in  $A^1$ ; this is related to the definition in [BD]. So  $\sigma_A(a)$  and  $\sigma_{A^1}(a)$  determine the same set, but sometimes we prefer to write  $\sigma_{A^1}(a)$  for exactness.

Let  $B$  is a subalgebra of  $A$ ; it is called a *spectral subalgebra* of  $A$  if  $\sigma_B(a) \setminus \{0\} = \sigma_A(a) \setminus \{0\}$  for every  $a \in B$ . Every ideal of  $A$  is a spectral subalgebra of  $A$ ; a normed algebra  $A$  is a  $Q$ -algebra  $\Leftrightarrow A$  is a spectral subalgebra of  $\widehat{A}$ , see [P, Theorem 4.2.10] and [KS, Lemma 20.9].

**1.2.5. The joint spectrum.** For the joint spectral theory in Banach algebras we refer to [H, M]. If  $M = (a_\alpha)_\Lambda \subset A$  is a family in  $A$  then the *left spectrum*  $\sigma^l(M)$  is the set of all families  $\lambda = (\lambda_\alpha)_\Lambda \subset \mathbb{C}$  such that the family  $M - \lambda := (a_\alpha - \lambda_\alpha)_\Lambda$  generates the proper left ideal of  $A^1$ . The *right spectrum*  $\sigma^r(M)$  is defined similarly by replacing ‘left’ by ‘right’;  $\sigma(M) = \sigma^l(M) \cup \sigma^r(M)$  is called the *Harte spectrum*, or simply the *spectrum*. We write  $\sigma^l(a)$  and  $\sigma^r(a)$  if  $M = \{a\}$ . Let  $A$  be unital. A subalgebra  $B$  of  $A$  is called *unital* if it contains the identity element of  $A$ , and *inverse-closed* if  $B$  contains  $x^{-1}$  for every  $x \in B$  which is invertible in  $A$ . If  $B$  is inverse-closed then  $\sigma_B(a) = \sigma_A(a)$  for every  $a \in B$ .

The following folklore lemma determines the operational possibilities of the joint spectra.

**Lemma 1.2.** *Let  $A$  be an algebra and  $M = (a_\alpha)_\Lambda$  be a family in  $A$ . Then*

- (1)  $\lambda \notin \sigma_A^l(M)$  if and only if  $\lambda|_N \notin \sigma_A^l(N)$  for some finite subfamily  $N \subset M$ ;
- (2) If  $B$  is a unital subalgebra of  $A^1$  and  $M \subset B$  then  $\sigma_A^l(M) \subset \sigma_B^l(M)$ ;
- (3) If  $f : A \rightarrow C$  is a surjective homomorphism then  $\sigma_C^l(f(M)) \subset \sigma_A^l(M)$ ;
- (4)  $\sigma^l(M) = \sigma^l(M/\text{rad}(A))$ .

*Proof.* (1), (2) and (4) are standard and trivial.

(3) Consider only the case when  $C$  is unital and  $A$  is not. One can extend  $f$  up to  $f' : A^1 \rightarrow C$  by setting  $f'(1_A) = 1_C$ , whence  $\sigma_C^l(f'(M)) \subset \sigma_A^l(M)$ , but  $f'(M) = f(M)$ .  $\square$

If  $N = (T_\beta)$  is a family in  $L(X)$  then  $\sigma_X^p(N)$ , or simply  $\sigma^p(N)$ , is a *point spectrum* of  $N$ , i.e., the set of all  $\lambda = (\lambda_\beta) \subset \mathbb{C}$  such that there is a non-zero vector  $\zeta \in X$  and  $T_\beta \zeta = \lambda_\beta \zeta$ ;  $\lambda$  is called an *eigenvalue* of  $N$  corresponding to *eigenvector*  $\zeta$ . If  $X$  is normed then it is useful to use an *approximate point spectrum*  $\sigma_X^a(N)$ , or simply  $\sigma^a(N)$ ; it is the set of all  $\lambda = (\lambda_\beta) \subset \mathbb{C}$  such that there is a net  $\zeta = (\zeta_\gamma) \subset X$  with  $\lim_\gamma \|\zeta_\gamma\| > 0$  and  $\lim_\gamma \|(T_\beta - \lambda_\beta)\zeta_\gamma\| = 0$  for every  $\beta$ ;  $\lambda$  is called an *approximate eigenvalue* of  $N$  corresponding to *approximate eigenvector*  $\zeta$ . If  $A$  is a normed algebra and  $M$  is a family in  $A$  then set  $\sigma_A^a(M) = \sigma_A^a(L_M)$ .

If  $T \in L(X)$  then put  $\sigma_X(T) := \sigma_{L(X)}(T)$ . If  $T$  is one-to-one and onto then  $T$  is invertible in  $L(X)$  whence  $\sigma_X(T) = \sigma_X^p(T) \cup \sigma_{L(X)}^r(T)$ . If  $X$  is a Banach space,  $T$  is bounded and the kernel and the image of  $T$  satisfy the mentioned conditions then  $T$  is invertible in  $\mathcal{B}(X)$  by Banach’s theorem, whence

$$\sigma_{\mathcal{B}(X)}(T) = \sigma_X^p(T) \cup \sigma_{\mathcal{B}(X)}^r(T) = \sigma_X^p(T) \cup \sigma_{L(X)}^r(T) = \sigma_X(T) \quad (1.12)$$

**Theorem 1.3.** *Let  $A$  be an algebra and  $M$  be a family in  $A$ . Then*



- (1)  $\sigma_A^l(M) = \cup_{\pi \in \text{Irr}(A^1)} \sigma_{X_\pi}^p(\pi(M))$ ;
- (2) If  $\pi$  is a representation of  $A^1$  by bounded operators on a normed space  $X$  then  $\sigma_X^a(\pi(M)) \subset \sigma_A^l(M)$ ;
- (3) If  $A$  is a  $Q$ -algebra then  $\sigma_A^l(M) = \cup_{\pi \in \text{Irr}_n(A^1)} \sigma_{X_\pi}^a(\pi(M))$ .

*Proof.* (1) For  $\lambda \in \sigma_A^l(M)$ , there is a maximal left ideal  $J$  of  $A^1$  such that  $M - \lambda = (a_\alpha - \lambda_\alpha) \subset J \neq A^1$ . The representation  $\pi$  of  $A^1$  defined by  $\pi(b) = L_b|_X$  is strictly irreducible where  $X = A^1/J$ . For  $\xi = 1/J \in X$  we obtain that  $\pi(a_\alpha)\xi = \lambda_\alpha\xi$  for every  $\alpha$ . Therefore  $\lambda \in \sigma_X^p(\pi(M))$ . We proved the inclusion  $\subset$ .

Conversely, let  $\lambda \in \sigma_{X_\pi}^p(\pi(M))$  for some  $\pi \in \text{Irr}(A^1)$ . Then  $\pi(a_\alpha - \lambda_\alpha)\xi = 0$  for some non-zero vector  $\xi \in X_\pi$  and each  $\alpha$ . But  $\{x \in A^1 : \pi(x)\xi = 0\}$  is a proper left ideal of  $A^1$ . Therefore  $\lambda \in \sigma_A^l(M)$ .

(2) Let  $0 \in \sigma_{X_\pi}^a(\pi(M)) \setminus \sigma_A^l(M)$ . By Lemma 1.2, there is a finite subfamily  $N = (a_\alpha)_{\Lambda'} \subset M$  with  $0 \notin \sigma_A^l(N)$ . So there are  $b_\alpha \in A^1$  for  $\alpha \in \Lambda'$  such that  $\sum_{\Lambda'} b_\alpha a_\alpha = 1$ . If  $\zeta = (\zeta_\gamma) \in X$  is an approximate eigenvector related to 0 then

$$0 < \lim_{\gamma} \|\zeta_\gamma\| \leq \lim_{\gamma} \sum_{\alpha \in \Lambda'} \|\pi(b_\alpha a_\alpha) \zeta_\gamma\| \leq \sum_{\alpha \in \Lambda'} \|\pi(b_\alpha)\| \lim_{\gamma} \|\pi(a_\alpha) \zeta_\gamma\| = 0,$$

a contradiction. Therefore  $\sigma_{X_\pi}^a(\pi(M)) \subset \sigma_A^l(M)$ .

(3) follows from (1) and (2).  $\square$

One can obtain the related statements for  $\sigma_A^r(M)$  if pass to the opposite algebra.

**1.2.6. Spectrum and primitive ideals.** If an algebra  $A$  is normed then  $\sigma(a)$  is not empty by the Gelfand-Mazur theorem; if  $A$  is a  $Q$ -algebra then  $\sigma(a)$  is a compact subset of  $\mathbb{C}$ .

**Theorem 1.4.** *Let  $A$  be an algebra and  $a \in A$ . Then*

- (1)  $\sigma_A(a) \setminus \sigma_A^l(a) \subset \cup_{\pi \in \text{Irr}(A^1)} \sigma_{L(X_\pi)}^r(\pi(a))$ ;
- (2) *The following chain of equalities holds:*

$$\begin{aligned} \sigma_A(a) &= \cup_{I \in \text{Prim}(A^1)} \sigma_{A^1/I}(a/I) = \cup_{\pi \in \text{Irr}(A^1)} \sigma_{\pi(A^1)}(\pi(a)) = \cup_{\pi \in \text{Irr}(A^1)} \sigma_{X_\pi}(\pi(a)) \\ &= \cup_{\pi \in \text{Irr}(A^1)} (\sigma_{X_\pi}^p(\pi(a)) \cup \sigma_{X_\pi}^r(\pi(a))) ; \end{aligned}$$

- (3) *If  $A$  is a  $Q$ -algebra then*

$$\sigma_A(a) = \cup_{\pi \in \text{Irr}_n(A^1)} \sigma_{\pi(A)}(\pi(a)) = \cup_{\pi \in \text{Irr}_n(A^1)} \sigma_{\mathcal{B}(X_\pi)}(\pi(a)) ;$$

- (4) *If  $A$  is a  $Q_b$ -algebra then*

$$\sigma_A(a) = \cup_{\pi \in \text{Irr}_b(A^1)} (\sigma_{X_\pi}^p(\pi(a)) \cup \sigma_{\mathcal{B}(X_\pi)}^r(\pi(a))) .$$

*Proof.* (1) Let  $\lambda \in \sigma_A(a) \setminus \sigma_A^l(a)$ ; then there is  $b \in A^1$  such that  $b(a - \lambda) = 1$ . Assume, to the contrary, that  $\lambda \notin \sigma_{L(X_\pi)}^r(\pi(a))$  for every  $\pi \in \text{Irr}(A)$ : there is an operator  $T_\pi$  on  $X_\pi$  such that  $(\pi(a) - \lambda)T_\pi = 1$ . As  $\pi(b)(\pi(a) - \lambda) = 1$  then  $T_\pi = \pi(b)$  for every  $\pi \in \text{Irr}(A^1)$ . Hence  $(a - \lambda)b - 1 \in \cap_{\pi \in \text{Irr}(A^1)} \ker \pi = \text{rad}(A^1)$  and  $(a - \lambda)b = 1 + c$  for some  $c \in \text{rad}(A^1)$ . So  $(a - \lambda)b(1 + c)^{-1} = 1$  and  $\lambda \notin \sigma_A(a)$ , a contradiction.

(2) By Lemma 1.2(2-3),

$$\begin{aligned} \sigma_A(a) &\supset \sigma_{A^1/I}(a/I) = \sigma_{\pi(A^1)}(\pi(a)) \supset \sigma_{L(X_\pi)}(\pi(a)) \\ &= \sigma_{X_\pi}^p(\pi(a)) \cup \sigma_{L(X_\pi)}^r(\pi(a)) \end{aligned}$$

where  $\pi \in \text{Irr}(A^1)$  and  $I = \ker \pi$ . The result follows from (1) and Theorem 1.3(1).

(3) If  $A$  is a  $Q$ -algebra then every  $\pi \in \text{Irr}(A^1)$  is equivalent to a (bounded) strictly irreducible representation by bounded operators on a normed space  $X_\pi$ . As  $\pi(A) \subset \overline{\pi(A)} \subset \mathcal{B}(X_\pi) \subset L(X_\pi)$  then

$$\sigma_{L(X_\pi)}(\pi(a)) = \sigma_{\mathcal{B}(X_\pi)}(\pi(a)) \subset \overline{\sigma_{\pi(A)}}(\pi(a)) \subset \sigma_{\pi(A)}(\pi(a))$$

for every  $\pi \in \text{Irr}(A^1)$ , and the result follows from (2).

(4) If  $A$  is a  $Q_b$ -algebra then every  $\pi \in \text{Irr}(A^1)$  is equivalent to a (bounded) strictly irreducible representation by bounded operators on a Banach space  $X_\pi$ . Then the result follows from (3) and (1.12).  $\square$

The first equality in Theorem 1.4(2) was proved in [Z2, Proposition 1] (for Banach algebras).

**Corollary 1.5.** *Let  $A$  be an algebra, and let  $I$  be an ideal of  $A$  and  $a \in A$ . Then*

(1) *If  $(A/I)^1 \cong A^1/I$  then*

$$\sigma_{A/I}(a/I) = \sigma_{A^1/\text{kh}(I; A^1)}(a/\text{kh}(I; A^1)) = \cup_{J \in \text{h}(I; A^1)} \sigma_{A^1/J}(a/J);$$

(2) *If  $(A/I)^1 \not\cong A^1/I$  then*

$$\sigma_{A/I}(a/I) = \sigma_{A/\text{kh}(I; A)}(a/\text{kh}(I; A)) = \cup_{J \in \text{h}(I; A)} \sigma_{A/J}(a/J);$$

(3) *If  $A$  is a  $Q$ -algebra then  $\sigma_{A/I}(a/I) = \sigma_{A/\bar{I}}(a/\bar{I})$ .*

*Proof.* (1) As  $\text{kh}(I; A^1) = q_I^{-1}(\text{rad}(A^1/I))$  then

$$A^1/\text{kh}(I; A^1) \cong (A^1/I) / \text{rad}(A^1/I) \cong ((A/I) / \text{rad}(A/I))^1 \quad (1.13)$$

The first equality follows from Lemma 1.2(4). As

$$\text{Prim}(A^1/I) = \{J/I : J \in \text{h}(I; A^1)\}$$

by (1.10) and  $(A^1/I) / (J/I) \cong A^1/J$  then, by Theorem 1.4(2),

$$\sigma_{A/I}(a/I) = \cup_{J \in \text{h}(I; A^1)} \sigma_{(A^1/I)/(J/I)}(a/(J/I)) = \cup_{J \in \text{h}(I; A^1)} \sigma_{A^1/J}(a/J).$$

(2) This is a case when  $A/I$  is unital and  $A$  is not unital. We see as in (1.13) that  $A/\text{kh}(I; A) = ((A/I) / \text{rad}(A/I))^1$  is unital and the first equality follows. Furthermore,  $\text{Prim}(A/I) = \{J/I : J \in \text{h}(I; A)\}$  by (1.10). If  $J \in \text{h}(I; A)$  then  $A/J$  is also unital, whence as in (1) we have  $\sigma_{A/I}(a/I) = \cup_{J \in \text{h}(I; A)} \sigma_{A/J}(a/J)$ .

(3) Consider only the case  $(A/I)^1 \cong A^1/I$  because the other case is similar. As primitive ideals are closed,  $\bar{I} \subset J$  for every  $J \in \text{Prim}(A^1)$  with  $I \subset J$ . Then  $I \subset \bar{I} \subset \text{kh}(I; A^1)$  implies  $\sigma_{A^1/\text{kh}(I; A^1)}(a/\text{kh}(I; A^1)) \subset \sigma_{A/\bar{I}}(a/\bar{I}) \subset \sigma_{A/I}(a/I)$  and the result follows from (1).  $\square$

**Remark 1.6.** *As a consequence of Corollary 1.5, the spectrum of  $a/I$  in the quotient  $A/I$  of a  $Q$ -algebra  $A$  by a possibly unclosed ideal  $I$  is a compact set in  $\mathbb{C}$ .*

**1.2.7. Banach ideals.** Let  $A$  be a normed algebra with norm  $\|\cdot\|$ ; we write also  $(A; \|\cdot\|)$ . An ideal  $I$  of  $A$  is called *normed* if there are a norm  $\|\cdot\|_I$  on  $I$  and  $t > 0$  such that

$$\|x\| \leq t \|x\|_I$$

for every  $x \in I$ ; the norm  $\|\cdot\|_I$  is called *flexible* if  $t = 1$ . A normed ideal  $I$  with a flexible norm is also called a flexible ideal;  $I$  is called a Banach ideal if it is complete with respect to  $\|\cdot\|_I$ .

An ideal of  $A$  with the norm inherited from  $A$  is of course flexible. We will describe now a more interesting class of examples.

It is well known that the sum of two closed ideals  $I$  and  $J$  of a Banach algebra  $A$  may be non-closed, see an excellent discussion in [D4]. One can consider this sum as a Banach ideal with the norm

$$\|a\|_{I+J} = \inf \{\|x\| + \|y\| : a = x + y, x \in J, y \in I\}$$

for every  $a \in J + I$ . If  $J \cap I = 0$  then there is only one pair  $x \in J, y \in I$  with  $a = x + y$  and  $\|a\|_{I+J} = \|x\| + \|y\|$ . So we have in general for  $x \in J, y \in I$  that

$$\|x + y\|_{I+J} = \inf \{\|x - z\| + \|y + z\| : z \in J \cap I\}.$$

It is clear that if  $a$  lies in  $I$  or  $J$  then  $\|a\| = \|a\|_{I+J}$ . Therefore  $J \cap I$  is a closed ideal of  $K := (J + I; \|\cdot\|_{I+J})$ . Then  $K/(J \cap I)$  is a Banach algebra with the norm

$$\|a/(J \cap I)\|_{K/(J \cap I)} = \|x/(J \cap I)\| + \|y/(J \cap I)\|$$

for  $a/(J \cap I) = x/(J \cap I) + y/(J \cap I)$  with  $x \in J, y \in I$ . Also,  $I/(J \cap I)$  is a closed ideal of  $K/(J \cap I)$ , whence clearly

$$\begin{aligned} \|a/I\|_{K/I} &= \|(a/(J \cap I))/(I/(J \cap I))\|_{(K/(J \cap I))/(I/(J \cap I))} \\ &= \inf \{\|x/(J \cap I)\| + \|(y - z)/(J \cap I)\| : z \in I\} \\ &= \|x/(J \cap I)\|. \end{aligned} \tag{1.14}$$

On the other hand, as  $I$  is a closed ideal of  $K$  then

$$\begin{aligned} \|a/I\|_{K/I} &= \inf \{\|a - z\|_{I+J} : z \in I\} \\ &= \inf \{\|x\| + \|y\| : a = x + y + z, x \in J, y, z \in I\}. \end{aligned}$$

So we obtain the following

**Lemma 1.7.** *Let  $(A, \|\cdot\|)$  be a normed algebra, and let  $I, J$  be closed ideals of  $A$ . Then*

(1)  *$J/(J \cap I)$  is isometrically isomorphic to  $(J + I)/I$  with the norm*

$$\|a/I\|_{(J+I)/I} = \inf \{\|x\| + \|y\| : a = x + y + z, x \in J, y, z \in I\}.$$

(2)  *$(J + I)/I$  with norm  $\|\cdot\|_{(J+I)/I}$  is a flexible ideal of  $A/I$ .*

*Proof.* (1) There is a bounded isomorphism  $\phi$  from  $J/(J \cap I)$  onto  $(J + I)/I$ , so for every  $x/(J \cap I)$  with  $x \in J$  there is only one  $a/I$  with  $a \in J + I$ . The above argument clearly works if  $A$  is a normed algebra, and (1.14) shows that  $\phi$  is an isometry.

(2) is obvious.  $\square$

**Remark 1.8.** *It follows from the lemma that if  $A$  is a Banach algebra then the ideal  $(J + I)/I$  with norm  $\|\cdot\|_{(J+I)/I}$  is a Banach ideal of  $A/I$ . Indeed,  $J/(J \cap I)$  is complete whence  $(J + I)/I$  with norm  $\|\cdot\|_{(J+I)/I}$  is complete.*

## 2. RADICALS AND OTHER IDEAL MAPS

**2.1. Definitions.** Radicals are defined on classes of algebras satisfying some natural conditions (see [ST5]). For our aims it is sufficient to assume as a rule that each of these classes,  $\mathfrak{U}$ , is either the class  $\mathfrak{U}_a$  of all associative complex algebras for the algebraic case or one of the classes  $\mathfrak{U}_b$  and  $\mathfrak{U}_n$  of all Banach and all normed associative complex algebras, respectively, for the topological case. We consider also radicals defined on subclasses of these classes, for example on the subclass of  $\mathfrak{U}_n$  consisting of all  $Q$ -algebras and the subclass of  $\mathfrak{U}_b$  consisting of all  $C^*$ -algebras.

We use the term *morphism* for a surjective homomorphism in the algebraic case and a open continuous surjective homomorphism in the topological case. In what follows, unless necessity for clarity we omit words ‘algebraic’ or ‘topological’ for

notions of the radical theory in statements valid in both the algebraic and topological contexts. Sometimes we add necessary topological or algebraic specifications in square brackets. For instance, the expression “a [closed] ideal” means that the ideal in the proposition must be closed when we consider the proposition in the topological context.

A map  $P$  on  $\mathfrak{U}$  is called a [closed] *ideal map* if  $P(A)$  is a [closed] ideal of  $A$  for each [normed]  $A \in \mathfrak{U}$ . An *algebraic* (respectively, *topological*) *radical* is a (respectively, closed) ideal map  $P$  that satisfies the following axioms:

**Axiom 1.**  $f(P(A)) \subset P(B)$  for every morphism  $f : A \longrightarrow B$  of algebras from  $\mathfrak{U}$ ;

**Axiom 2.**  $P(A/P(A)) = 0$ ;

**Axiom 3.**  $P(P(A)) = P(A)$ ;

**Axiom 4.**  $P(I)$  is an ideal of  $A$  contained in  $P(A)$ , for each ideal  $I \in \mathfrak{U}$  of  $A$ .

It is assumed in Axiom 4 that if  $A$  is normed then the norm on  $I$  is inherited from  $A$ . If a (closed) ideal map  $P$  satisfies Axiom 1 on  $\mathfrak{U}$  then  $P$  is called a *pre-radical* on  $\mathfrak{U}$ ;  $P$  is called *pliant* if Axiom 1 on  $\mathfrak{U}$  holds with algebraic morphisms. A (non-necessarily closed) ideal map  $P$  is a *preradical with topological morphisms* if it defined on a class of normed algebras and satisfies Axiom 1 with topological morphisms (this class of maps contains the restrictions of arbitrary algebraic preradicals to classes of normed algebras). Algebraic preradicals are always pliant.

**Lemma 2.1.** *Let  $P$  be a pliant preradical, and let  $A$  be an algebra with two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$ . Then  $P(A; \|\cdot\|_1) = P(A; \|\cdot\|_2)$ .*

*Proof.* Indeed, the identity map  $(A; \|\cdot\|_1) \longrightarrow (A; \|\cdot\|_2)$  is an algebraic isomorphism.  $\square$

Sometimes the intermediate definitions are useful: a preradical  $P$  on  $\mathfrak{U} \subset \mathfrak{U}_n$  is called *strict* if

$$\overline{f(P(A))} = P(B) \quad (2.1)$$

for any continuous isomorphism  $f : A \longrightarrow B$  of algebras from  $\mathfrak{U}$ , and called *strong* if  $P$  satisfies Axiom 1 with respect to the continuous surjective homomorphisms of algebras from  $\mathfrak{U}$ . It is clear that every strict preradical is strong.

**Remark 2.2.** *For radicals on  $\mathfrak{U}_b$  or  $\mathfrak{U}_a$  our definition coincides with Dixon's one, but for normed algebras the definitions differ:*

- (1) *We do not assume that ideals in Axiom 4 are closed;*
- (2) *In [D3] the morphisms are not necessarily open.*

A preradical  $P$  is called *hereditary* if

$$P(I) = I \cap P(A)$$

for every ideal  $I$  of an algebra  $A \in \mathfrak{U}$ . The class of hereditary radicals is especially important; the heredity of a preradical  $P$  implies the fulfillment of Axioms 3 and 4 for  $P$ . So one can define *hereditary radicals* as hereditary preradicals satisfying Axiom 2.

In several cases we will impose on a preradical  $P$ , defined on  $\mathfrak{U}_b$ , a more strong condition – the *condition of Banach heredity*:

$$P(L, \|\cdot\|_L) = L \cap P(A) \text{ for any Banach ideal } (L, \|\cdot\|_L) \subset A, \quad (2.2)$$

where  $A$  is a Banach algebra. One can introduce similarly the *condition of flexible heredity*, for normed algebras.

An algebra  $A \in \mathfrak{U}$  is called  *$P$ -radical* if  $A = P(A)$  and  *$P$ -semisimple* if  $P(A) = 0$ . Let  $\mathbf{Rad}(P)$  denote the class of all  $P$ -radical algebras and  $\mathbf{Sem}(P)$  the class of all

$P$ -semisimple algebras. A preradical  $P$  is called *uniform* if every subalgebra  $B \in \mathfrak{U}$  of a  $P$ -radical algebra is  $P$ -radical. It is clear that uniform radicals are hereditary.

By definition [D3], an *under radical* is a preradical which satisfies all axioms besides, possibly, Axiom 2. For instance, any hereditary preradical is an under radical; moreover, restrictions of algebraic radicals to appropriate classes of normed algebras are algebraic under radicals in general. The dual notion for under radicals is a notion of an over radical. By definition [D3], an *over radical* is a preradical which satisfies all axioms besides, possibly, Axiom 3.

In the class of preradicals on  $\mathfrak{U}$  there is a natural order:  $P_1 \leq P_2$  if  $P_1(A) \subset P_2(A)$  for each  $A \in \mathfrak{U}$ . We write  $P_1 < P_2$  if  $P_1 \leq P_2$  and  $P_1 \neq P_2$ . It is easy to see that

$$P_1 \leq P_2 \implies \mathbf{Rad}(P_1) \subset \mathbf{Rad}(P_2) \text{ and } \mathbf{Sem}(P_2) \subset \mathbf{Sem}(P_1). \quad (2.3)$$

The following theorem is useful for comparing radicals.

**Theorem 2.3.** *Let  $P_1$  and  $P_2$  be over radicals on  $\mathfrak{U}$ , and let  $R_1$  and  $R_2$  be under radicals on  $\mathfrak{U}$ . Then*

- (1)  $\mathbf{Sem}(P_2) \subset \mathbf{Sem}(P_1)$  if and only if  $P_1 \leq P_2$ ;
- (2)  $\mathbf{Rad}(R_1) \subset \mathbf{Rad}(R_2)$  if and only if  $R_1 \leq R_2$ .

*Proof.* Let  $A$  be an algebra. Then  $R_1(A) \in \mathbf{Rad}(R_1)$  by Axiom 3 and  $A/P_2(A) \in \mathbf{Sem}(P_2)$  by Axiom 2.

- (1) If  $\mathbf{Sem}(P_2) \subset \mathbf{Sem}(P_1)$  then  $A/P_2(A) \in \mathbf{Sem}(P_1)$ . As  $P_1$  is a preradical,

$$q(P_1(A)) \subset P_1(A/P_2(A)) = 0$$

for the standard quotient map  $q : A \longrightarrow A/P_2(A)$ . Therefore  $P_1(A) \subset P_2(A)$ , i.e.,  $P_1 \leq P_2$ , and (2.3) completes the proof.

- (2) If  $\mathbf{Rad}(R_1) \subset \mathbf{Rad}(R_2)$  then  $R_1(A) \in \mathbf{Rad}(R_2)$ . As  $R_1(A)$  is an ideal of  $A$ ,

$$R_1(A) = R_2(R_1(A)) \subset R_2(A)$$

by Axiom 4, i.e.,  $R_1 \leq R_2$ , and the converse follows by (2.3).  $\square$

**Corollary 2.4.** *Let  $P_1$  and  $P_2$  be radicals on  $\mathfrak{U}$ . If  $\mathbf{Sem}(P_1) = \mathbf{Sem}(P_2)$  or  $\mathbf{Rad}(P_1) = \mathbf{Rad}(P_2)$  then  $P_1 = P_2$ .*

## 2.2. Classes of algebras.

**2.2.1. Base classes of algebras.** Let us say more about classes of algebras on which the radicals are defined. It is convenient to assume by default that a class of algebras contains all appropriate [topologically] isomorphic images of its elements.

A class  $\mathfrak{U}$  of algebras is called *algebraically universal* if it contains quotients and ideals of algebras from  $\mathfrak{U}$ . A class  $\mathfrak{U}$  of normed algebras is called *universal* if it contains quotients by closed ideals and ideals of algebras from  $\mathfrak{U}$ , and *ground* if it contains quotients by closed ideals and closed ideals of algebras from  $\mathfrak{U}$ .

For instance,  $\mathfrak{U}_a$  is algebraically universal,  $\mathfrak{U}_n$  is universal, and  $\mathfrak{U}_b$  is ground. These classes are main in this paper. We also use the following classes of algebras:

- (1)  $\mathfrak{U}_q$  and  $\mathfrak{U}_{q_b}$  are the class of all  $Q$ -algebras and  $Q_b$ -algebras, respectively. These classes are universal [ST5].
- (2)  $\mathfrak{U}_b^u$  is the smallest universal class containing all Banach algebras. An algebra  $A$  is called a *subideal* of an algebra  $B$  if there is a finite series of algebras  $A = I_0 \subset \cdots \subset I_n = B$  such that  $I_{i-1}$  is an ideal of  $I_i$  for  $i = 1, \dots, n$ ; in such a case  $A$  is called an *n-subideal* of  $B$ . By [ST5, Theorem 2.24],  $\mathfrak{U}_b^u$  is the class of all subideals of Banach algebras.

- (3)  $\mathfrak{U}_{c^*}$  is the class of all  $C^*$ -algebras. Recall that every closed ideal of a  $C^*$ -algebra is automatically self-adjoint [Dm, Proposition 1.8.2]. So this class is ground.

2.2.2. *About the definition of radicals in  $C^*$ -algebras.* We will also consider the restrictions of radicals of Banach algebras to  $\mathfrak{U}_{c^*}$ . The work with  $C^*$ -algebra has several advantages:

- (1) The sum of two closed ideals of a  $C^*$ -algebra is closed [Dv, Corollary 1.5.6];
- (2) Each irreducible  $*$ -representation of a  $C^*$ -algebra is strictly irreducible [Dm, Theorem 2.8.3];
- (3) A closed ideal of a closed ideal of a  $C^*$ -algebra is an ideal of the algebra [Dm, Proposition 1.8.5].

So Axiom 4 for radicals on  $C^*$ -algebras is equivalent to the following:

**Axiom 4 for  $\mathfrak{U}_{c^*}$ .**  $P(I) \subset P(A)$  for any closed ideal  $I$  of  $A$ .

As  $C^*$ -algebras are semisimple, every morphism in  $\mathfrak{U}_{c^*}$  is automatically continuous by Johnson's theorem [J]. It seems to be natural to consider all  $*$ -epimorphisms (i.e.  $*$ -morphisms) as morphisms in  $\mathfrak{U}_{c^*}$ . The following result shows that this does not change the class of preradicals.

**Theorem 2.5.** *Let  $P$  be a closed ideal map on  $\mathfrak{U}_{c^*}$ . If  $f(P(A)) \subset P(B)$  for each  $*$ -morphism  $f : A \rightarrow B$  then the same is true for all morphisms.*

*Proof.* Let  $f : A \rightarrow B$  be an epimorphism,  $J = \ker f$  and  $C = A/J$ . Then  $f = g \circ q_J$ , where  $g$  is an isomorphism of  $C$  onto  $B$ . So it suffices to show that  $q_J(P(A)) \subset P(C)$  and  $g(P(C)) \subset P(B)$ . The first inclusion is evident because  $q_J$  is a  $*$ -epimorphism.

One can consider  $B$  as an operator  $C^*$ -algebra, that is, a closed  $*$ -subalgebra of  $\mathcal{B}(H)$  where  $H$  is a Hilbert space. It was proved by T. Gardner [Gr] that  $g = g_1 \circ g_2$  where  $g_2$  is a  $*$ -isomorphism of  $C$  onto  $B$ , and  $g_1 : B \rightarrow B$  acts by the rule

$$g_1(x) = vxv^{-1}$$

where  $v$  is an invertible positive operator on  $H$ . So, to prove that  $g(P(C)) \subset P(B)$ , it suffices to show that  $g_1$  preserves closed ideals of  $B$ .

Let  $S$  be the operator on  $\mathcal{B}(H)$  defined by  $Sx = vxv^{-1}$  for all  $x \in \mathcal{B}(H)$ . By the assumption,  $S$  preserves  $B$  and  $g_1 = S|_B$ . Clearly  $S$  is a product of two commuting operators:  $S = L_v R_{v^{-1}}$ . Since  $L_v$  and  $R_{v^{-1}}$  have positive spectra on  $\mathcal{B}(H)$ , the same is true for  $S$  on  $\mathcal{B}(H)$ .

Let us denote by  $\log$  the holomorphic extension of the function

$$\phi(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (z-1)^k$$

from the disk  $D = \{z : |z-1| < 1\}$  to  $\mathbb{C} \setminus (-\infty, 0]$ . By the functional calculus, this function can be applied to every element of a Banach algebra whose spectrum doesn't intersect  $(-\infty, 0] \subset \mathbb{R}$ . As  $E := \{K \in \mathcal{B}(\mathcal{B}(H)) : KB \subset B\}$  is a Banach algebra and  $S \in E$  has spectrum outside of  $(-\infty, 0] \subset \mathbb{R}$  then the operator  $T = \log S$  preserves  $B$ .

Moreover,  $T = \log L_v + \log R_{v^{-1}}$ . Indeed, to prove that

$$\begin{aligned} \log(1 + \lambda(L_v - 1))(1 + \lambda(R_{v^{-1}} - 1)) &= \log(1 + \lambda(L_v - 1)) \\ &\quad + \log(1 + \lambda(R_{v^{-1}} - 1)) \end{aligned}$$

for  $\lambda = 1$ , it suffices to show that the equality holds for sufficiently small  $|\lambda|$ . Clearly the equality  $\phi(KL) = \phi(K) + \phi(L)$  is checked for commuting operators  $K, L$  having spectra in  $D$  by the calculation as in the case of numerical series.

Let  $u = \log v$ . As  $0 = \log 1 = \log v + \log v^{-1}$  and  $\log L_v = L_{\log v}$  then  $\log L_v = L_u$  and  $\log R_{v^{-1}} = -R_u$ . We proved that the operator  $T$  acts by the formula

$$Tx = ux - xu$$

for  $x \in \mathcal{B}(H)$  and preserves  $B$ .

The restriction of  $T$  to  $B$  is a derivation of  $B$  and therefore preserves closed ideals of  $B$ . Indeed, if  $I$  is a closed ideal of  $B$  then it is a  $C^*$ -algebra, so it has a bounded approximate identity whence, by [BD, Theorem 11.10], each  $a \in I$  can be written in the form  $a = bc$  for  $a, b \in I$ , and

$$T(a) = bT(c) + T(b)c \in I.$$

Therefore  $S = \exp(T)$  also preserves all closed ideals of  $B$ .  $\square$

As a result, Axiom 1 for radicals on  $C^*$ -algebras is equivalent to the following:

**Axiom 1 for  $\mathfrak{U}_c^*$ .**  $f(P(A)) \subset P(B)$  for any  $*$ -morphism  $f : A \longrightarrow B$ .

### 2.3. Some important examples.

2.3.1. *The Jacobson radical.* The most famous radical on  $\mathfrak{U}_a$  is the *Jacobson radical*  $\text{rad}$  which is defined by

$$\text{rad}(A) = \ker \text{Prim}(A^1) := \cap \{I : I \in \text{Prim}(A^1)\}$$

for every algebra  $A$ ;  $\text{rad}$  is an algebraic hereditary radical. As usual,  $A$  is called *radical* if  $A = \text{rad}(A)$ , and *semisimple* if  $\text{rad}(A) = 0$ . By Johnson's theorem [J], the topology of a complete norm in a semisimple Banach algebra is unique.

**Theorem 2.6.** *Let  $A, B$  be Banach algebras, and let  $P$  be a topological over radical such that  $\text{rad} \leq P$  on Banach algebras. Then*

- (1) *If  $f : A \longrightarrow B$  is an algebraic morphism then  $f(P(A)) \subset P(B)$ ;*
- (2) *If  $f : A \longrightarrow B$  is an algebraic isomorphism then  $f(P(A)) = P(B)$ .*

*Proof.* (1) Let  $B' = B/\text{rad}(B)$  and  $q : B \longrightarrow B'$  be the standard quotient map. Then  $q \circ f : A \longrightarrow B'$  is a topological morphism by Johnson's result [A, Corollary 5.5.3]. Therefore  $(q \circ f)(P(A)) \subset P(B')$ , whence  $f(P(A)) \subset q^{-1}(P(B'))$ .

Let  $g : B/\text{rad}(B) \rightarrow B/P(B)$  be the natural morphism, then

$$g(P(B/\text{rad}(B))) \subset P(B/P(B)) = 0$$

whence  $P(B') = P(B/\text{rad}(B)) \subset \ker g = P(B)/\text{rad}(B) = q(P(B))$ . Thus

$$f(P(A)) \subset q^{-1}(q(P(B))) = P(B).$$

(2) follows from (1) since  $f^{-1} : B \longrightarrow A$  is also an algebraic morphism.  $\square$

As a consequence, topological radicals larger than (or equal to)  $\text{rad}$  on Banach algebras don't depend on the complete norm topology.

Let  $\text{Rad}$  be the restriction of  $\text{rad}$  to Banach algebras:  $\text{Rad} = \text{rad}|_{\mathfrak{U}_b}$ . Then  $\text{Rad}$  is a pliant hereditary topological radical, while the restriction  $\text{rad}|_{\mathfrak{U}_n}$  is not even a topological radical [D3]. The reason was already discussed: a normed algebra can have nonclosed primitive ideals. It is clear that  $\text{Rad}$  has a unique hereditary extension to  $\mathfrak{U}_b^a$ , namely  $\text{rad}|_{\mathfrak{U}_b^a}$ . The hereditary extension of  $\text{Rad}$  to normed algebras is given by the *regular Jacobson radical*  $\text{Rad}^r$  (see Section 3.3):

$$\text{Rad}^r(A) = \left\{ a \in A : \rho(ab) = 0 \ \forall b \in \widehat{A} \right\} \quad (2.4)$$

where  $\rho(a)$  is a (geometric) spectral radius  $\inf_n \|a^n\|^{1/n}$ . Below we consider the other topological hereditary extensions of  $\text{Rad}$  to  $\mathfrak{U}_n$ .

**2.3.2. Primitive maps and related radicals.** Let  $\mathfrak{U}$  be a base class of algebras. A rule that indicates, for each algebra  $A \in \mathfrak{U}$ , a subset  $\Omega(A)$  of  $\text{Prim}(A)$  is called a *primitive map* on  $\mathfrak{U}$  if

- (1<sub>pm</sub>)  $\Omega(B) = \{f(I) : I \in \Omega(A)\}$  for an injective morphism  $f : A \rightarrow B$  of algebras from  $\mathfrak{U}$ ;
- (2<sub>pm</sub>)  $\Omega(J) = \{J \cap I : I \in \Omega(A), J \cap I \neq J\}$  for every ideal  $J \in \mathfrak{U}$  of  $A$ ;
- (3<sub>pm</sub>)  $\Omega(A/J) = \{J/I : I \in \Omega(A), J \subset I\}$  for every [closed] ideal  $J \in \mathfrak{U}$  of  $A$ .

If (1<sub>pm</sub>) holds for arbitrary isomorphisms of algebras  $A, B \in \mathfrak{U}$ , then we say that  $\Omega$  is *pliant*; in other words, a primitive map  $\Omega$  on  $\mathfrak{U}$  is pliant if the following condition holds:

- (4<sub>pm</sub>)  $\Omega(A)$  doesn't depend on the choice of a norm in  $A$ , for every  $A$  from  $\mathfrak{U}$  (in short: on the choice of  $\mathfrak{U}$ -norm).

Clearly all primitive maps on  $\mathfrak{U}_a$  and on  $\mathfrak{U}_{c^*}$  are pliant.

**Proposition 2.7.** *All primitive maps defined on Banach algebras are pliant.*

*Proof.* Let  $A$  be a Banach algebra. It follows from (3<sub>pm</sub>) that  $\Omega(A)$  is uniquely determined by  $\Omega(A/\text{Rad}(A))$ , but  $A/\text{Rad}(A)$  has only one complete norm topology by Johnson's theorem.  $\square$

The following proposition is straightforward.

**Proposition 2.8.** *A primitive map defined on some class  $\mathfrak{U}$  of normed algebras is pliant if and only if it does not depend on the choice of a  $\mathfrak{U}$ -norm on semisimple algebras from  $\mathfrak{U}$ .*

It is easy to see from the results of Section 1.2.3 that setting  $\Omega(A) = \text{Prim}(A)$  for each  $A$ , we obtain a primitive map on  $\mathfrak{U}_a$ ; it will be denoted by  $\text{Prim}$ .

**Theorem 2.9.** *Let  $\mathcal{F} = (\Omega_\alpha)$  be a family of (pliant) primitive maps. Then*

- (1)  $\Omega_{\cap \mathcal{F}} : A \rightarrow \cap_\alpha \Omega_\alpha(A)$  and  $\Omega_{\cup \mathcal{F}} : A \rightarrow \cup_\alpha \Omega_\alpha(A)$  are (pliant) primitive maps;
- (2) If  $\Omega_i$  is a (pliant) primitive map for  $i = 1, 2$  then so is  $\Omega_1 \setminus \Omega_2 : A \mapsto \Omega_1(A) \setminus \Omega_2(A)$ .

*Proof.* (1) Since

$$\begin{aligned} \cup_\alpha \Omega_\alpha(J) &= \cup_\alpha \{J \cap I : I \in \Omega_\alpha(A), J \cap I \neq J\} \\ &= \{J \cap I : I \in \cup_\alpha \Omega_\alpha(A), J \cap I \neq J\} \end{aligned}$$

we obtain (2<sub>pm</sub>) for  $\Omega_{\cup \mathcal{F}}$ . The remaining assertions can be proved similarly.

(2). Since

$$\begin{aligned} (\Omega_1 \setminus \Omega_2)(A/J) &= \{J/I : I \in \Omega_1(A), J \subset I\} \setminus \{J/I : I \in \Omega_2(A), J \subset I\} \\ &= \{J/I : I \in \Omega_1(A) \setminus \Omega_2(A), J \subset I\} \end{aligned}$$

we get (3<sub>pm</sub>) for  $\Omega_1 \setminus \Omega_2$ . The other conditions can be checked in a similar way.  $\square$

As usual for subsets of  $\text{Prim}(A)$ ,  $\ker \Omega(A)$  is simply  $\cap \{I : I \in \Omega(A)\}$ . We define an ideal map  $\Pi_\Omega$  by

$$\Pi_\Omega(A) = \ker \Omega(A)$$

for every  $A \in \mathfrak{U}$ , meaning that  $\Pi_\Omega(A) = A$  if  $\Omega(A) = \emptyset$ , or, more formally, by  $\Pi_\Omega(A) = A \cap \ker \Omega(A)$ . Not all primitive maps determine radicals (see Example 2.14). The obstacle is that the ideals  $\Pi_\Omega(A)$  can be non-closed.

**Theorem 2.10.** *Let  $\Omega$  be a primitive map on  $\mathfrak{U}$ . Then*



- (1) If  $(\Omega$  is pliant and)  $\mathfrak{U} \subset \mathfrak{U}_q$  then  $\Pi_\Omega$  is a (pliant) topological hereditary radical;
- (2) If  $\Omega$  is pliant and  $\mathfrak{U}_q \subset \mathfrak{U} \subset \mathfrak{U}_n$  then  $\Pi_\Omega$  is a pliant hereditary preradical;
- (3) If  $\mathfrak{U} = \mathfrak{U}_a$  then  $\Pi_\Omega$  is an algebraic hereditary radical.

*Proof.* (3) Let  $A \in \mathfrak{U}$  be an algebra, and let  $J \in \mathfrak{U}$  be an ideal of  $A$ . It follows from  $(2_{pm})$  that  $\Pi_\Omega(J) = J \cap \Pi_\Omega(A)$ . So  $\Pi_\Omega$  is hereditary.

Let  $J = \Pi_\Omega(A)$ . Then  $J \subset I$  for every  $I \in \Omega(A)$ . By (1.4),

$$\Pi_\Omega(A/J) = \ker \Omega(A)/J = \Pi_\Omega(A)/J = J/J = 0.$$

Since every morphism is represented as the superposition of a standard quotient map and an injective morphism,  $(1_{pm})$  and  $(3_{pm})$  imply that  $\Pi_\Omega$  is a pliant radical on  $\mathfrak{U}$ .

(1) Indeed, every strictly irreducible representation of a normed  $Q$ -algebra  $A$  is equivalent to a bounded representation by bounded operators on a normed space [ST5, Theorem 2.1]. So  $\text{Prim}(A)$  consists of closed ideals and  $\Pi_\Omega$  is a closed ideal map on normed  $Q$ -algebras. As in (3),  $\Pi_\Omega$  is a hereditary radical.

(2) is evident.  $\square$

**Remark 2.11.** An algebra  $A \in \mathfrak{U}$  is  $\Pi_\Omega$ -radical if and only if  $\Omega(A) = \emptyset$ , and  $A$  is  $\Pi_\Omega$ -semisimple if and only if  $\bigcap \{I : I \in \Omega(A)\} = 0$ ; it is clear that every  $\Pi_\Omega$ -semisimple algebra is rad-semisimple, i.e.  $\text{rad} \leq \Pi_\Omega$  on  $\mathfrak{U}$ . In particular, Theorem 2.6 holds for  $P = \Pi_\Omega$ .

The following statement is obvious.

**Corollary 2.12.** Let  $\Omega$  be a pliant primitive map on a universal class  $\mathfrak{U}$  of normed algebras. Then  $\Pi_\Omega$  satisfies the condition of flexible heredity (in particular, the condition of Banach heredity) and  $f(P(A)) = P(B)$  for an algebraic injective morphism  $f : A \rightarrow B$  of any algebras  $A, B \in \mathfrak{U}$ .

See the definitions of  $\text{Prim}_n(A)$  and  $\text{Prim}_b(A)$  for a normed algebra  $A$  in Section 1.2.3.

**Example 2.13.**  $\text{Prim}_n$  and  $\text{Prim}_b$  are primitive maps on  $\mathfrak{U}_n$ .

*Proof.* In virtue of Section 1.2.3, it is sufficient to check that the declared properties of representations from  $\text{Irr}_n(A)$  or  $\text{Irr}_b(A)$  are preserved under passing to an ideals, quotients, actions of isomorphisms and taking the converse manipulations. But this is straightforward in virtue of [ST5, Lemmas 2.3 and 2.4].  $\square$

The topological radicals  $\Pi_{\text{Prim}_n}$  and  $\Pi_{\text{Prim}_b}$  coincide with the introduced in [D3, ST5] extensions  $\text{rad}_b$  and  $\text{rad}_n$  of  $\text{Rad}$  to normed algebras. On the base of  $\text{Prim}_n$  and  $\text{Prim}_b$  one can construct other useful primitive maps by imposing additional requirements on representations  $\pi$  from  $\text{Irr}_n(A)$  or  $\text{Irr}_b(A)$ .

**Example 2.14.** By Theorem 2.9,  $\Omega_n := \text{Prim} \setminus \text{Prim}_n$  and  $\Omega_b := \text{Prim} \setminus \text{Prim}_b$  are primitive maps on  $\mathfrak{U}_n$ , but  $\Pi_{\Omega_n}$  and  $\Pi_{\Omega_b}$  are not closed ideal maps on  $\mathfrak{U}_n$ ;  $Q$ -algebras are  $\Pi_{\Omega_n}$ -radical, and  $\Pi_{\Omega_n}(A)$  is a  $Q$ -algebra for every normed algebra  $A$ .

The simplest characteristic of a representation is its dimension. Recall that the dimension of  $\pi \in \text{Irr}(A)$  is the dimension of  $X_\pi$ . Let  $N$  be a subset of  $\mathbb{N} \cup \{\infty\}$ , and let  $\text{Irr}^{\dim \in N}(A)$  be the set of all representations  $\pi \in \text{Irr}(A)$  such that  $\dim X_\pi \in N$ . Respectively, let  $\text{Prim}^{\dim \in N}(A) = \left\{ \ker \pi : \pi \in \text{Irr}^{\dim \in N}(A) \right\}$ . The simplest special cases of the relation  $\dim \in N$  are  $\dim = 1$ ,  $\dim > 1$ ,  $\dim < \infty$ .

**Theorem 2.15.** Let  $N \subset \mathbb{N} \cup \{\infty\}$ . Then  $\text{Prim}^{\dim \in N}$  is a pliant hereditary primitive map.

*Proof.* It suffices to note that the representation spaces are not changed when we extend a representation from an ideal to the algebra, restrict to an ideal or induce the strictly irreducible representations from a quotient.  $\square$

**Corollary 2.16.** *Let  $N \subset \mathbb{N} \cup \{\infty\}$ . Then the map*

$$\text{rad}^{\dim \in N} : A \longmapsto \Pi_{\text{Prim}^{\dim \in N}}(A)$$

*is a pliant hereditary radical.*

Note that  $\text{rad}^{\dim \in N}$  is a pliant hereditary topological radical on  $Q$ -algebras.

**Example 2.17.** *An algebra  $A$  is  $\text{rad}^{\dim > 1}$ -radical  $\Leftrightarrow A$  admits only one-dimensional strictly irreducible representations, and  $A$  is  $\text{rad}^{\dim = 1}$ -semisimple  $\Leftrightarrow$  the intersection of kernels of one-dimensional strictly irreducible representations is trivial.*

*Respectively,  $A$  is  $\text{rad}^{\dim = \infty}$ -radical  $\Leftrightarrow A$  admits only finite-dimensional strictly irreducible representations, and  $A$  is  $\text{rad}^{\dim < \infty}$ -semisimple  $\Leftrightarrow$  the intersection of kernels of finite-dimensional strictly irreducible representations is trivial.*

Let us finish by showing that in terms of primitive maps one can define one of the most important objects of the theory of  $C^*$ -algebras — the largest GCR-ideal.

Recall that a  $C^*$ -algebra  $A$  is called *elementary* if it is isomorphic to the algebra  $\mathcal{K}(H)$  of all compact operators on some Hilbert space  $H$ . Furthermore,  $A$  is a *CCR-algebra* (a *GCR-algebra*) if for every irreducible representation  $\pi$  of  $A$ , the image,  $\pi(A)$ , consists of compact operators (respectively, contains a non-zero compact operator, or, equivalently, all compact operators). A simplest example of a CCR-algebra is  $\mathcal{K}(H)$ .

For  $A \in \mathfrak{U}_{c^*}$ , let  $\Omega_{\text{gcr}}(A)$  denote the set of all  $I \in \text{Prim}(A)$  such that  $A/I$  has no elementary ideals.

**Theorem 2.18.**  $\Pi_{\Omega_{\text{gcr}}}$  is a topological radical on  $\mathfrak{U}_{c^*}$ ;  $\Pi_{\Omega_{\text{gcr}}}(A)$  is the largest GCR-ideal for every  $A \in \mathfrak{U}_{c^*}$ .

*Proof.* It is easy to check that  $\Omega_{\text{gcr}}$  is a primitive map whence  $\Pi_{\Omega_{\text{gcr}}}$  is a topological radical on  $\mathfrak{U}_{c^*}$ .

Let  $A$  be a  $C^*$ -algebra and  $J_0 = \Pi_{\Omega_{\text{gcr}}}(A) = \bigcap \{I \in \Omega_{\text{gcr}}(A)\}$ . To show that  $J_0$  is a GCR-algebra, take an irreducible representation  $\pi$  of  $J_0$  on a Hilbert space  $H$  and extend it to an irreducible representation (again  $\pi$ ) of  $A$  on  $H$ . Then  $\ker \pi \notin \Omega_{\text{gcr}}(A)$ , so  $\pi(A)$  contains an elementary ideal  $K$ . Since  $K$  is irreducible and  $\mathcal{K}(H)$  is a CCR-algebra,  $K = \mathcal{K}(H)$ . Then

$$0 \neq \pi(J_0)K(H) \subset \pi(J_0)\pi(A) \subset \pi(J_0),$$

so  $\pi(J_0)$  contains a non-zero compact operator. This means that  $J_0$  is a GCR-algebra.

Let now  $J$  be a GCR-ideal of  $A$ . If  $I \in \Omega_{\text{gcr}}(A)$  then  $I = \ker \pi$  for an irreducible representation  $\pi$  of  $A$  such that  $\pi(A)$  has no elementary ideals. But the restriction of  $\pi$  to  $J$  is also irreducible (if non-zero), so  $\pi(J)$  must contain  $\mathcal{K}(H)$ . Therefore  $\mathcal{K}(H) \subset \pi(A)$ , a contradiction. Thus  $\pi(J) = 0$  and then  $J \subset I$ . So

$$J \subset \bigcap \{I : I \in \Omega_{\text{gcr}}(A)\} = J_0.$$

$\square$

Another way to establish that the map sending each  $C^*$ -algebra  $A$  to its largest GCR-ideal is a hereditary topological radical, will be considered in Section 2.3.7.

2.3.3. *The tensor Jacobson radical  $\mathcal{R}_t$ .* It is defined via the tensor spectral radius. Let  $A$  be a normed algebra,  $M = (a_i)_1^\infty$  and  $N = (b_j)_1^\infty$  be *summable* families in  $A$ , i.e.  $\|M\|_+ = \sum_i \|a_i\| < \infty$  and  $\|N\|_+ = \sum_i \|b_i\| < \infty$ . The norm of a family does not depend on the order, so one can consider the corresponding equivalence relation  $\simeq$  between families (see details in [ST6]). Define the product  $MN$  within up to  $\simeq$  as a family  $H = (c_k)_1^\infty$  where  $c_k = a_i b_j$  for  $k = \phi(i, j)$  and  $\phi$  is an arbitrary bijection of  $\mathbb{N} \times \mathbb{N}$  onto  $\mathbb{N}$ . Then the power  $M^{n+1}$  is defined by  $M^{n+1} \simeq MM^n$  for every  $n > 0$  and the *tensor (spectral) radius*  $\rho_t(M)$  is defined by

$$\rho_t(M) = \inf_n \|M^n\|_+^{1/n} = \lim_{n \rightarrow \infty} \|M^n\|_+^{1/n}.$$

Let us define by  $N \sqcup M$  the disjunct union of families  $M = (a_i)$  and  $N = (b_j)$  (that is a family equivalent to  $(b_1, a_1, b_2, a_2, \dots)$ ); then  $\mathcal{R}_t(A)$  is defined as the set of all  $a \in A$  such that  $\rho_t(\{a\} \sqcup M) = \rho_t(M)$  for every summable family  $M$  in  $A$ ;  $\mathcal{R}_t$  is a uniform topological radical [ST5, ST6].

The term *tensor* is justified by the fact that  $a \in \mathcal{R}_t(A)$  if and only if  $a \otimes b \in \text{Rad}(A \widehat{\otimes} B)$  for every normed algebra  $B$  and  $b \in B$  [ST6, Theorem 3.36]. By [ST6, Theorem 3.29],

$$\rho_t(M) = \rho_t(M/\mathcal{R}_t(A)). \quad (2.5)$$

There is a problem whether  $\mathcal{R}_t = \text{Rad}$  on Banach algebras. In the algebraic case it is known that the tensor product of a radical algebra and the other algebra can be not radical.

**Problem 2.19.** *Is there an algebraic radical  $P$  on  $\mathfrak{U}_a$  such that  $P = \mathcal{R}_t$  on  $\mathfrak{U}_n$ ?*

2.3.4. *The compactly quasinilpotent radical  $\mathcal{R}_{cq}$ .* Let  $A$  be a normed algebra, and let  $M \subset A$  be bounded. Define the *norm*  $\|M\| = \sup_{a \in M} \|a\|$  and the *joint spectral radius*

$$\rho(M) = \inf_n \|M^n\|^{1/n} = \lim_{n \rightarrow \infty} \|M^n\|^{1/n}$$

where  $M^n = \{a_1 \cdots a_n : a_i \in M\}$  [RS]. Let  $\mathfrak{k}(A)$  be the set of all precompact subsets of  $A$ . Then  $\mathcal{R}_{cq}(A)$  is the set of all  $a \in A$  such that  $\rho(\{a\} \cup M) = \rho(M)$  for every  $M \in \mathfrak{k}(A)$ ;  $\mathcal{R}_{cq}$  is a uniform topological radical [ST5].

A normed algebra  $A$  is called *compactly quasinilpotent* if  $\rho(M) = 0$  for every  $M \in \mathfrak{k}(A)$ ; the completion of compactly quasinilpotent algebra is again compactly quasinilpotent [ST5, Lemma 4.11]. Note that  $\mathcal{R}_{cq}(A)$  is the largest compactly quasinilpotent ideal [ST5, Corollary 4.21].

An important fact related to this radical is that every compact operator  $a$  in the Jacobson radical of a closed operator algebra  $A$  generates a compactly quasinilpotent ideal of  $A$ ; a close argument was a key in the proof that  $A$  has a hyperinvariant subspace if  $a \neq 0$  [S]. The other invariant subspace results [T2, ST1] are connected with the calculation of the joint spectral radius. It was established in [ST1] that for a precompact set  $M$  of compact operators on a Banach space  $X$

$$\rho(M) = r(M) \quad (2.6)$$

where the *Berger-Wang radius*  $r(M)$  is defined by [BW]:

$$r(M) = \limsup_{n \rightarrow \infty} \sup \{\rho(a) : a \in M^n\}^{1/n}.$$

We call (2.6) the *Berger-Wang formula* because the matrix version of (2.6), for  $\dim X < \infty$ , was established in [BW]. A useful contribution in the calculation of  $\rho$  is the equality

$$\rho(M) = \rho(M/\mathcal{R}_{cq}(A)) \quad (2.7)$$

for every  $M \in \mathfrak{k}(A)$  [ST5, Theorem 4.18].

There is a version of the joint spectral radius for bounded countable families  $M = (a_n)_1^\infty$  in  $A$ , where  $M^n$  is calculated by rules of families [ST6].

**2.3.5. The hypocompact radical  $\mathcal{R}_{\text{hc}}$ .** Let  $A$  be a normed algebra. An element  $a \in A$  is called *compact* if  $W_a := L_a R_a$  (see Section 1.2.3) is a compact operator on  $A$ ;  $A$  is *compact* if it consists of compact elements, *bicompact* if  $L_a R_b$  is a compact operator on  $A$  for every  $a, b \in A$ , and *hypocompact* if every non-zero quotient of  $A$  by a closed ideal has a non-zero compact element. Then  $\mathcal{R}_{\text{hc}}(A)$  is defined as the largest hypocompact ideal of  $A$  (equivalently, as the smallest closed ideal  $J$  of  $A$  such that  $A/J$  has no non-zero compact elements);  $\mathcal{R}_{\text{hc}}$  is a hereditary topological radical [ST7].

This radical plays an important role in the theory of the joint spectral radius. It was proved in [ST7] that the following equality (an *algebra version of the joint spectral radius formula*):

$$\rho(M) = \max\{\rho(M/\mathcal{R}_{\text{hc}}(A)), r(M)\} \quad (2.8)$$

holds for every normed algebra  $A$  and  $M \in \mathfrak{k}(A)$ . In particular, it follows from (2.8) that

$$\mathcal{R}_{\text{hc}}(A) \cap \text{Rad}^r(A) \subset \mathcal{R}_{\text{cq}}(A) \quad (2.9)$$

for every normed algebra  $A$ . It should be noted that  $A \mapsto \mathcal{R}_{\text{hc}}(A) \cap \text{Rad}^r(A)$  is also a hereditary topological radical on normed algebras. This radical is called the *Jacobson hypocompact radical* and is denoted by  $\mathcal{R}_{\text{jhc}}$ .

If  $A = \mathcal{B}(X)$  for a Banach space  $X$ , then  $\mathcal{R}_{\text{hc}}(A) \supset \mathcal{K}(X)$ , so that (2.8) implies (2.6) and a stronger result, the *operator version of the joint spectral radius formula*:

$$\rho(M) = \max\{\rho_e(M), r(M)\} \quad (2.10)$$

that holds for every  $M \in \mathfrak{k}(\mathcal{B}(X))$ , where the *essential spectral radius*  $\rho_e(M)$  is defined as the joint spectral radius  $\rho(M/\mathcal{K}(X))$  in the Calkin algebra  $\mathcal{B}(X)/\mathcal{K}(X)$  (see details in [ST7]).

There is the largest of topological radicals  $P$  which can change  $\mathcal{R}_{\text{hc}}$  in (2.8) [ST7]; it is denoted by  $\mathcal{R}_{\text{bw}}$  and called the *Berger-Wang radical*.

A normed algebra  $A$  is called a *Berger-Wang algebra* if (2.6) holds for every  $M \in \mathfrak{k}(A)$ .

**2.3.6. The hypofinite radicals  $\mathcal{R}_{\text{hf}}$  and  $\mathfrak{R}_{\text{hf}}$ .** We begin with the algebraic hypofinite radical  $\mathfrak{R}_{\text{hf}}$  and describe its construction more transparently because of the lack of references. An element  $a$  of an algebra  $A$  is called a *finite rank element* if  $\dim aAa < \infty$ . An algebra  $A$  is called *finite* if it consists of finite rank elements, *bifinite* if  $\dim aAb < \infty$  for every  $a, b \in A$ , and *hypofinite* if  $A/I$  contains a non-zero finite rank element for every ideal  $I \neq A$ . We transfer these notions to ideals. Let  $F(A)$  be the set of all finite rank elements of  $A$ .

**Theorem 2.20.** *Let  $A$  be an algebra, and let  $J$  be an ideal of  $A$ . Then*

- (1)  $f(F(A)) \subset F(B)$  for every morphism  $f : A \rightarrow B$ ;
- (2) If  $F(J) \neq 0$  then  $J \cap F(A) \neq 0$ ;
- (3) If  $x \in F(A)$  and  $I$  is an ideal generated by  $x$  then  $L_a R_b$  is a finite rank operator on  $A$  for every  $a, b \in I$ ;
- (4) If  $A$  is hypofinite and  $I$  a non-zero (one-sided) ideal then  $I \cap F(A) \neq 0$ ;
- (5) The following conditions are equivalent:
  - (a)  $J$  is a hypofinite ideal of  $A$ ;
  - (b) for every morphism  $f : A \rightarrow B$ , either  $f(J) = 0$  or  $f(J) \cap F(B) \neq 0$ ;
  - (c) there is an increasing transfinite sequence  $(J_\alpha)_{\alpha \leq \gamma}$  of ideals of  $A$  such that  $J_0 = 0$ ,  $J_\gamma = J$ ,  $J_\alpha = \cup_{\alpha' < \alpha} J_{\alpha'}$  for every limit ordinal  $\alpha$ , and all  $J_{\alpha+1}/J_\alpha$  are bifinite.
- (6) The following conditions are equivalent:
  - (a)  $A$  is hypofinite;
  - (b) all ideals and quotients of  $A$  are hypofinite;

- (c)  $A/I$  and  $I$  are hypofinite for some ideal  $I$  of  $A$ .
- (7) There is a largest hypofinite ideal of  $A$  (which is denoted by  $\mathfrak{R}_{\text{hf}}(A)$ );
- (8)  $\mathfrak{R}_{\text{hf}}(J) = J \cap \mathfrak{R}_{\text{hf}}(A)$ ;
- (9)  $A/\mathfrak{R}_{\text{hf}}(A)$  has no non-zero hypofinite ideals and finite rank elements.

*Proof.* (1)  $W_{f(a)}B = f(W_aA)$  is finite-dimensional if  $a \in F(A)$ .

(2) It is clear that

$$W_{ba} = L_b W_a R_b = R_a W_b L_a \quad (2.11)$$

for every  $a, b \in A$ . If  $a \in F(J)$  then  $W_{ba} \in F(A)$ , whence  $JF(J) \subset J \cap F(A)$ . If  $JF(J) = 0$  then clearly  $F(J) \subset J \cap F(A)$ .

(3) follows from (2.11).

(4) Assume that  $I$  is a left ideal. Let  $K = \{a \in A : aI = 0\}$ ;  $K$  is an ideal of  $A$ . If  $K = A$  then  $I \subset F(A)$ , otherwise  $A/K$  has a non-zero finite rank element  $b$ . Let  $q_K : A \rightarrow A/K$  be the standard quotient map and  $q_K(a) = b$ . Clearly there is  $x \in J$  such that  $ax \neq 0$ , it is easy to check that  $ax \in F(A)$ .

(5a)  $\implies$  (5b) Let  $I = \ker f$  and  $K = I \cap J$ . Assume that  $f(J) \neq 0$ ; then there is a non-zero  $a \in J/K \cap F(A/K)$  by (2). Let  $b \in A$  be such that  $a = q_K(b)$ ; then  $0 \neq f(b) \in F(B)$ .

(5b)  $\implies$  (5c) Assume that we already have built  $J_\beta$  by transfinite induction. If  $J \neq J_\beta$  then  $J/J_\beta$  has a non-zero finite rank element by definition, and contains a non-zero finite rank element  $a$  of  $A$  by (2). Then  $J/J_\beta$  contains a non-zero bifinite ideal  $I$  of  $A$  by (3). Take the preimage of  $I$  in  $A$  as  $J_{\beta+1}$ . This proves the implication.

(5c)  $\implies$  (5a) Let  $I$  be an ideal of  $J$  and  $I \neq J$ . Then there is the first ordinal  $\alpha$  such that  $J_\alpha$  doesn't lie in  $I$ . Then  $q_I(J_\alpha) \subset F(J/I)$ .

(6a)  $\implies$  (6b) It is clear that the quotients of a hypofinite algebra are hypofinite. Let  $K$  be an ideal of  $A$ ,  $f : A \rightarrow B$  a morphism and  $I = \ker f$ . Assume that  $f(K) \neq 0$ ; then  $q_I(K) \neq 0$ . As  $A$  is hypofinite then there is  $a \in K$  such that  $0 \neq q_I(a) \in F(A/I)$  by (4). Then  $0 \neq f(a) \in F(B)$  by (1). Therefore  $K$  is hypofinite by (5).

(6b)  $\implies$  (6c) is obvious, and (6c)  $\implies$  (6a) follows by (5): one can build a sequence of ideals for  $A$  as in (5c) having such sequences for  $I$  and  $A/I$ .

(7) Let  $K$  be the sum of all hypofinite ideals of  $A$ . Then an arbitrary morphism  $f : A \rightarrow B$  vanishes on  $K$  if and only if it vanishes on each hypofinite ideal of  $A$ . Therefore  $K$  is hypofinite by (5).

(8) As  $J \cap \mathfrak{R}_{\text{hf}}(A)$  is a hypofinite ideal of  $J$ , we obtain that  $J \cap \mathfrak{R}_{\text{hf}}(A) \subset \mathfrak{R}_{\text{hf}}(J)$ . Let  $I$  be the ideal of  $A$  generated by  $\mathfrak{R}_{\text{hf}}(J)$ . Then clearly  $I^3 \subset \mathfrak{R}_{\text{hf}}(J)$  is hypofinite and  $I/I^3$  is bifinite, whence  $I$  is hypofinite by (6). Therefore  $\mathfrak{R}_{\text{hf}}(J) \subset I \subset \mathfrak{R}_{\text{hf}}(A)$ .

(9) If  $I$  is a non-zero hypofinite ideal of  $A/\mathfrak{R}_{\text{hf}}(A)$  then its preimage  $K$  in  $A$  is hypofinite by (6), whence  $\mathfrak{R}_{\text{hf}}(A) \subsetneq I$ , a contradiction. Therefore  $A/\mathfrak{R}_{\text{hf}}(A)$  has no non-zero finite rank elements.  $\square$

**Corollary 2.21.**  $\mathfrak{R}_{\text{hf}}$  is a hereditary radical.

*Proof.* By Theorem 2.20(8-9), it suffices to show that  $\mathfrak{R}_{\text{hf}}$  is an algebraic preradical. Let  $f : A \rightarrow B$  be a morphism, and let  $q : B \rightarrow B/\mathfrak{R}_{\text{hf}}(B)$  be the standard quotient map. Then  $q \circ f$  is a morphism  $A \rightarrow B/\mathfrak{R}_{\text{hf}}(B)$  and it follows from Theorem 2.20(5b) that either  $(q \circ f)(\mathfrak{R}_{\text{hf}}(A)) = 0$  or  $B/\mathfrak{R}_{\text{hf}}(B)$  has a non-zero finite rank element which is impossible by Theorem 2.20(9). Therefore  $(q \circ f)(\mathfrak{R}_{\text{hf}}(A)) = 0$  and  $f(\mathfrak{R}_{\text{hf}}(A)) \subset \mathfrak{R}_{\text{hf}}(B)$ .  $\square$

The radical  $\mathfrak{R}_{\text{hf}}$  is called an (*algebraic*) *hypofinite radical*.

The radical  $\mathcal{R}_{\text{hf}}$  is a topological hereditary radical and can be defined as the largest closed-hypofinite ideal: a normed algebra (or ideal)  $A$  is called (*closed-*)

*hyppofinite* if every non-zero quotient of  $A$  by a *closed* ideal has a non-zero finite rank element of  $A$ , and *approximable* [ST6] if  $A = \overline{F(A)}$ . Elements of  $\overline{F(A)}$  are called *approximable elements* of  $A$ . Theorem 2.20 and Corollary 2.21 hold for  $\mathcal{R}_{\text{hf}}$  if one replaces the terms for  $\mathfrak{A}_{\text{hf}}$  by appropriate terms for  $\mathcal{R}_{\text{hf}}$ .

**2.3.7. Radicals on  $C^*$ -algebras.** Let  $P$  be a radical. It is known ([ST5, Theorems 2.9 and 2.10]) that the class  $\mathbf{Rad}(P)$  is stable under extensions and quotients, and that the [closure of] the union of an up-directed net of ideals  $I_\lambda \in \mathbf{Rad}(P)$  of some algebra also belongs to  $\mathbf{Rad}(P)$ . Moreover, if  $P$  is hereditary then each ideal of  $A \in \mathbf{Rad}(P)$  clearly belongs to  $\mathbf{Rad}(P)$ . The following result shows that the converse holds for  $C^*$ -algebras: a class of  $C^*$ -algebras with the listed above properties is  $\mathbf{Rad}(P)$  for some (hereditary) radical  $P$ .

**Theorem 2.22.** *Let  $\mathfrak{C}$  be a class of  $C^*$ -algebras containing  $*$ -isomorphic images of its elements and having the following properties:*

- (a) *If  $A \in \mathfrak{C}$  then  $A/J \in \mathfrak{C}$  for every closed ideal  $J$  of  $A$ ;*
- (b) *If  $J \in \mathfrak{C}$  is an ideal of a  $C^*$ -algebra  $A$  and  $A/J \in \mathfrak{C}$  then  $A \in \mathfrak{C}$ ;*
- (c) *If  $(J_\alpha)_\alpha \subset \mathfrak{C}$  is an up-directed net of ideals of a  $C^*$ -algebra  $A$  then  $\overline{\bigcup_\alpha J_\alpha} \in \mathfrak{C}$ .*

*Then*

- (1) *Each  $C^*$ -algebra  $A$  has a largest ideal  $R_{\mathfrak{C}}(A) \in \mathfrak{C}$ ;*
- (2)  *$R_{\mathfrak{C}}$  is a radical on  $\mathfrak{U}_{c^*}$  such that  $\mathbf{Rad}(R_{\mathfrak{C}}) = \mathfrak{C}$ ;*
- (3) *The radical  $R_{\mathfrak{C}}$  is hereditary if and only if  $\mathfrak{C}$  satisfies the condition*  
  - (d) *Any closed ideal of an algebra  $A \in \mathfrak{C}$  belongs to  $\mathfrak{C}$ ;*
- (4)  *$R_{\mathfrak{C}}$  is uniform if and only if  $B \in \mathfrak{C}$  for any  $C^*$ -subalgebra  $B$  of every  $A \in \mathfrak{C}$ .*

*Proof.* (1) If ideals  $I, J$  of  $A$  belong to  $\mathfrak{C}$  then

$$(I + J)/I = J/(I \cap J) \in \mathfrak{C}$$

whence  $I + J \in \mathfrak{C}$  by (a).

Let  $\Lambda$  be the set of all finite families of ideals of  $A$  that belong to  $\mathfrak{C}$ ; it is ordered by the inclusion. For  $\omega \in \Lambda$ , let  $J_\omega$  be the sum of all ideals in  $\omega$ . Then  $(J_\omega)_{\omega \in \Lambda} \subset \mathfrak{C}$  is an up-directed net of ideals of  $A$ , whence the closure of their sum is also in  $\mathfrak{C}$  by (c). Clearly it is a largest ideal in  $A$  that belongs to  $\mathfrak{C}$ .

(2) If  $f : A \rightarrow B$  is a  $*$ -morphism and  $J \in \mathfrak{C}$  is an ideal of  $A$  then  $f(J)$  is an ideal of  $B$   $*$ -isomorphic to a quotient of  $J$ . Hence  $f(J) \in \mathfrak{C}$  is an ideal of  $B$  and therefore  $f(J) \subset R_{\mathfrak{C}}(B)$ .

In particular,  $f(R_{\mathfrak{C}}(A)) \subset R_{\mathfrak{C}}(B)$ . We proved that the map  $A \rightarrow R_{\mathfrak{C}}(A)$  is a preradical.

As  $R_{\mathfrak{C}}(A) \in \mathfrak{C}$  then  $R_{\mathfrak{C}}(R_{\mathfrak{C}}(A)) = R_{\mathfrak{C}}(A)$ .

If  $I$  is a closed ideal of  $A$  then  $R_{\mathfrak{C}}(I)$  belongs to  $\mathfrak{C}$ , whence  $R_{\mathfrak{C}}(A) \subset I \cap R_{\mathfrak{C}}(A)$ .

Let  $N = R_{\mathfrak{C}}(A/R_{\mathfrak{C}}(A))$  and  $q = q_{R_{\mathfrak{C}}(A)}$ . Setting  $J = q^{-1}(N)$ , we have that  $J$  is a closed ideal of  $A$  and  $J/R_{\mathfrak{C}}(A) = q(J) = N$  whence  $J \in \mathfrak{C}$  by (b). Therefore  $J \subset R_{\mathfrak{C}}(A)$  and  $N = 0$ .

We proved that the map  $R_{\mathfrak{C}}$  is a radical on  $\mathfrak{U}_{c^*}$ ; the equality  $\mathbf{Rad}(R_{\mathfrak{C}}) = \mathfrak{C}$  is obvious.

(3) If  $R_{\mathfrak{C}}$  is hereditary then for each closed ideal  $I$  of an algebra  $A \in \mathfrak{C}$ , we have that  $R_{\mathfrak{C}}(I) = I \cap R_{\mathfrak{C}}(A) = I \cap A = I$ , whence  $I \in \mathfrak{C}$ .

Conversely, let  $\mathfrak{C}$  contain all closed ideals of all algebras in  $\mathfrak{C}$ . Since  $R_{\mathfrak{C}}(A) \in \mathfrak{C}$  then the ideal  $I \cap R_{\mathfrak{C}}(A)$  of  $R_{\mathfrak{C}}(A)$  belongs to  $\mathfrak{C}$  whence  $I \cap R_{\mathfrak{C}}(A) \subset R_{\mathfrak{C}}(I)$ . We proved that  $R_{\mathfrak{C}}$  is hereditary.

(4) obviously follows from (2). □

The above theorem permits to construct important examples of  $C^*$ -radicals. Recall that a  $C^*$ -algebra  $A$  is called a *GCR-algebra* (or  $C^*$ -algebra of type I) if  $\pi(A)$  contains a non-zero compact operator for any irreducible representation  $\pi$  of  $A$ . Furthermore  $A$  is *nuclear* if, for each  $C^*$ -algebra  $B$ , there is only one  $C^*$ -norm on the algebraic tensor product  $A \otimes B$ . If  $A$  contains an up-directed net of finite-dimensional  $*$ -subalgebras with dense union then  $A$  is called an *AF-algebra*.

**Corollary 2.23.** *Each  $C^*$ -algebra  $A$  has the largest GCR-ideal  $\mathcal{R}_{\text{gcr}}(A)$ , the largest nuclear ideal  $\mathcal{R}_{\text{nc}}(A)$  and the largest AF-ideal  $\mathcal{R}_{\text{af}}(A)$ . The maps  $\mathcal{R}_{\text{gcr}}$ ,  $\mathcal{R}_{\text{nc}}$  and  $\mathcal{R}_{\text{af}}$  are hereditary topological radicals on  $\mathfrak{U}_{c^*}$ .*

*Proof.* We should check the properties (a)-(d) for the corresponding classes. For GCR-algebras they are well known, as well as the existence of the largest GCR-ideal (see [Dm, Section 4.3]).

Clearly a quotient of an AF-algebra is an AF-algebra. The proof of the fact that a closed ideal of an AF-algebra is an AF-algebra see in [Dv, Lemma III.4.1]. The property (b) for AF-algebras was established by Brown [Br]. The property (c) for AF-algebras is obvious.

For nuclear algebras, the proof of (a), (b) and (d) can be found in [Tk2, Corollary 15.3.4]. The property (c) follows immediately from the fact that an inductive limit of nuclear  $C^*$ -algebras is nuclear; it was established in the first paper on the subject [Tk1].  $\square$

The radical  $\mathcal{R}_{\text{gcr}}$  is uniform, this follows from [Dm, Proposition 4.3.5]. The radicals  $\mathcal{R}_{\text{nc}}$  and  $\mathcal{R}_{\text{af}}$  are not uniform; the construction of a non-nuclear  $C^*$ -subalgebra of a nuclear  $C^*$ -algebra can be found in [Ws, Example 2.4]; for an example of a non-AF  $C^*$ -subalgebra of AF-algebra one can take an irrational rotation algebra (see [Dv, Theorem 6.5.2]).

### 3. PROCEDURES

We understand procedures in the theory of radicals as transforms of ideal maps. In other words, a *procedure* is a mapping  $P \mapsto P'$  from a class of ideal maps that act on a class  $\mathfrak{U}$  of algebras, into another class of ideal maps acting on a possibly different class  $\mathfrak{U}'$  of algebras.

**3.1. The convolution and superposition procedures.** Let  $A$  be a [normed] algebra. Increasing and decreasing transfinite sequences  $(I_\alpha)$  and  $(J_\alpha)$ , respectively, of [closed] ideals of  $A$  are called *transfinite chains* if

$$\begin{aligned} I_\alpha &= \cup_{\alpha' < \alpha} I_{\alpha'} & [I_\alpha &= \overline{\cup_{\alpha' < \alpha} I_{\alpha'}}], \\ J_\alpha &= \cap_{\alpha' < \alpha} J_{\alpha'} \end{aligned} \tag{3.1}$$

for every limit ordinal  $\alpha$ . The quotients  $I_{\alpha+1}/I_\alpha$  and  $J_\alpha/J_{\alpha+1}$  are called the *gap-quotients* of  $(I_\alpha)$  and  $(J_\alpha)$ , respectively.

Let  $P$  and  $R$  be preradicals satisfying Axiom 4. For each  $A \in \mathfrak{U}$ , we define transfinite chains of ideals  $(I_\alpha)$  and  $(J_\alpha)$  by the conditions

$$\begin{aligned} I_0 &= 0, & I_{\alpha+1} &= q_{I_\alpha}^{-1}(R(A/I_\alpha)), \\ J_0 &= A, & J_{\alpha+1} &= P(J_\alpha), \end{aligned}$$

respectively, for every ordinal  $\alpha$ , where  $q_{I_\alpha} : A \rightarrow A/I_\alpha$  is the standard quotient map.

Let  $R_\alpha(A) = I_\alpha$  and  $P_\alpha(A) = J_\alpha$  for every  $\alpha$ . Transfinite sequences  $(R_\alpha)$  and  $(P_\alpha)$  of ideal maps are called the *convolution* and *superposition chains* for  $R$  and

$P$ , respectively. The transfinite chains of ideals stabilize at some steps  $\delta = \delta(A)$  and  $\gamma = \gamma(A)$ :

$$I_\delta = I_{\delta+1} \quad \text{and} \quad J_\gamma = J_{\gamma+1}.$$

It was proved in [D3, Theorem 6.6 and Theorem 6.10] that the map  $R^*$  defined by

$$R^*(A) := I_\delta$$

satisfies Axiom 2, and the map  $P^\circ$  defined by

$$P^\circ(A) := J_\gamma$$

satisfies Axiom 3.

The maps  $R \mapsto R^*$  and  $P \mapsto P^\circ$  are called the *convolution* and *superposition procedures*, respectively. We underline that the convolution procedure for algebraic preradicals on  $\mathfrak{U}_a$  differs from the one for topological preradicals on  $\mathfrak{U}_n$  because the latter includes the operation of the closure. So we have two different — *algebraic* and *topological* — convolution procedures.

The following theorem was proved by Dixon [D3, Corollary 6.12] for algebraic (over/under) radicals and for topological (over/under) radicals on normed algebras in assumption that ideals in Axiom 4 are closed, in particular for (over/under) radicals on Banach algebras. We extend the Dixon results to preradicals of normed algebras satisfying Axiom 4 without any restrictions.

**Theorem 3.1.** *Let  $R, P$  be preradicals satisfying Axiom 4. Then*

- (1)  *$R^*$  is a smallest over radical larger than or equal to  $R$ .  
If  $R$  is an under radical then  $R^*$  is a radical;*
- (2)  *$P^\circ$  is a largest under radical smaller than or equal to  $P$ .  
If  $P$  is an over radical then  $P^\circ$  is a radical.*

*Proof.* Using the transfinite induction, Dixon proved [D3] that if  $R$  is an under radical and  $P$  is an over radical then all  $R_\alpha$  and  $P_\alpha$  are under and over radicals, respectively, under his assumption on ideals. It is easy to see from Dixon's proof that if  $R$  and  $P$  are preradicals satisfying Axiom 4 restricted to closed ideals then so are  $R_\alpha$  and  $P_\alpha$  for every  $\alpha$ .

Now let  $R$  and  $P$  satisfy Axiom 4. We have to check Axiom 4 for all maps from  $(R_\alpha)$  and  $(P_\alpha)$ ; then  $R^*$  and  $P^\circ$  will also satisfy Axiom 4. For other assertions of this theorem we refer to [D3].

Let  $A$  be a normed algebra, and let  $K$  be an ideal of  $A$ . Using the transfinite induction, we fix an ordinal  $\alpha$  and assume that  $(R_{\alpha'})_{\alpha' < \alpha}$  and  $(P_{\alpha'})_{\alpha' < \alpha}$  consist of preradicals satisfying Axiom 4.

If  $\alpha$  is a limit ordinal then all  $R_{\alpha'}(K) \subset R_{\alpha'}(A)$  and  $P_{\alpha'}(K) \subset P_{\alpha'}(A)$  are ideals of  $A$  for  $\alpha' < \alpha$ , whence

$$\begin{aligned} R_\alpha(K) &= \overline{\cup_{\alpha' < \alpha} R_{\alpha'}(K)}^{(K)} \subset \overline{\cup_{\alpha' < \alpha} R_{\alpha'}(A)}^{(A)} = R_\alpha(A), \\ P_\alpha(K) &= \cap_{\alpha' < \alpha} P_{\alpha'}(K) \subset \cap_{\alpha' < \alpha} P_{\alpha'}(A) = P_\alpha(A) \end{aligned} \quad (3.2)$$

are ideals of  $A$ . Indeed, to check this for  $R_\alpha(K)$  note that  $I := \cup_{\alpha' < \alpha} R_{\alpha'}(K)$  is an ideal of  $A$ , and for every  $x \in R_\alpha(K)$  there is a sequence  $(y_n) \subset I$  such that  $y_n \rightarrow x$  as  $n \rightarrow \infty$ . For every  $a \in A$ , one has  $ay_n, y_na \in I$  for each  $n$ , whence

$$ax, xa \in K \cap \overline{I}^{(A)} = \overline{I}^{(K)} = R_\alpha(K).$$

Therefore  $R_\alpha(K)$  is an ideal of  $A$ ; the same assertion for  $P_\alpha$  is trivial.

Let now  $\alpha = \alpha' + 1$ , for some ordinal  $\alpha'$ . Since  $P_{\alpha'}(K) \subset P_{\alpha'}(A)$  is an ideal of  $A$  then  $P_\alpha(K) = P(P_{\alpha'}(K))$  is an ideal of  $A$ . As  $P_{\alpha'}(K)$  is an ideal of  $P_{\alpha'}(A)$ , we obtain that

$$P_\alpha(K) = P(P_{\alpha'}(K)) \subset P(P_{\alpha'}(A)) = P_\alpha(A).$$



By the hypothesis of induction (valid for any  $A$ ),  $J := R_{\alpha'}(K) \subset R_{\alpha'}(A)$  is an ideal of  $A$ . Then, as  $K$  is an ideal of  $\overline{K}$ ,

$$J \subset R_{\alpha'}(\overline{K}) \quad (3.3)$$

is an ideal of  $\overline{K}$ . Since  $J = K \cap \overline{J}$ , the map  $x/J \mapsto q_{\overline{J}}(x)$  is an injective map from  $K/J$  onto the ideal  $q_{\overline{J}}(K)$  of  $A/\overline{J}$ . So

$$R_{\alpha}(K) = \{x \in K : q_{\overline{J}}(x) \in R(q_{\overline{J}}(K))\} \quad (3.4)$$

whence  $R_{\alpha}(K)$  is an ideal of  $A$ . As  $q_{\overline{J}}(K)$  is an ideal of  $\overline{K}/\overline{J}$ , we obtain that

$$R(q_{\overline{J}}(K)) \subset R(\overline{K}/\overline{J}). \quad (3.5)$$

Since  $\overline{J} \subset R_{\alpha'}(\overline{K})$  is an ideal of  $\overline{K}$  by (3.3), it follows that

$$q(R(\overline{K}/\overline{J})) \subset R(\overline{K}/R_{\alpha'}(\overline{K})) = R_{\alpha}(\overline{K})/R_{\alpha'}(\overline{K}). \quad (3.6)$$

for the natural map  $q : \overline{K}/\overline{J} \rightarrow \overline{K}/R_{\alpha'}(\overline{K})$ . It follows from (3.4), (3.5) and (3.6) that  $R_{\alpha}(K) \subset R_{\alpha}(\overline{K})$ . As  $\overline{K}$  is a closed ideal of  $A$  then  $R_{\alpha}(\overline{K}) \subset R_{\alpha}(A)$  by [D3, Theorem 6.6]. Therefore  $R_{\alpha}(K) \subset R_{\alpha}(A)$ . This completes the proof.  $\square$

We outline a part of the proof of Theorem 3.1 in the form of a separate statement.

**Proposition 3.2.** *Let  $R, R_1, P, P_1$  be preradicals satisfying Axiom 4. Then*

- (1) *If  $R_2$  is defined on  $\mathfrak{A}$  by  $R_2(A) = q_{R_1(A)}^{-1}(R(A/R_1(A)))$  then  $R_2$  is a preradical satisfying Axiom 4;*
  - (a) *If  $R, R_1$  are under radicals then  $R_2$  is an under radical;*
- (2) *If  $P_2$  is defined on  $\mathfrak{A}$  by  $P_2(A) = P(P_1(A))$  then  $P_2$  is a preradical satisfying Axiom 4;*
  - (a) *If  $P, P_1$  are over radicals then  $P_2$  is an over radical.*

*Proof.* This is the step  $\alpha \mapsto \alpha + 1$  of the transfinite induction applied in Theorem 3.1.  $\square$

**Theorem 3.3.** *Let  $R$  be an under radical, and let  $P$  be an over radical. Then*

- (1)  $\mathbf{Sem}(R^*) = \mathbf{Sem}(R)$ ;
- (2)  $\mathbf{Rad}(P^\circ) = \mathbf{Rad}(P)$ .

*Proof.* Let  $A$  be an algebra, and let  $(R_\alpha)$  and  $(P_\alpha)$  be the convolution and superposition chains of  $R$  and  $P$ , respectively.

- (1) If  $R(A) = 0$  then  $R_1(A) = R(A) = 0$ , whence  $R^*(A) = 0$ .
- (2) If  $A = P(A)$  then  $P_1(A) = P(A) = A$ , whence  $P^\circ(A) = A$ .  $\square$

**Theorem 3.4.** *Let  $R, R_1$  be under radicals, and let  $P, P_1$  be over radicals. Then*

- (1) *If  $R_1 \leq R$  then  $R_1^* \leq R^*$ ;*
- (2) *If  $P_1 \leq P$  then  $P_1^\circ \leq P^\circ$ ;*
- (3) *If  $R \leq P$  then  $R^* \leq P^\circ$ ;*
- (4) *If  $P \leq R$  then  $P^\circ \leq R^*$ .*

*Proof.* (1)  $R^*$  is a radical and  $R_1 \leq R \leq R^*$ . As  $R_1^*$  is the smallest radical among radicals  $R'$  such that  $R_1 \leq R'$  then  $R_1^* \leq R^*$ .

(2) is similar to (1):  $P_1^\circ$  is a radical and  $P_1^\circ \leq P_1 \leq P$ . As  $P^\circ$  is the largest radical among radicals  $P'$  such that  $P' \leq P$  then  $P_1^\circ \leq P^\circ$ .

(3) This is [D3, Theorem 6.11], but the following simple observation somewhat shortens the proof: as  $R \leq P$ , we have that

$$\mathbf{Sem}(P) \subset \mathbf{Sem}(R) = \mathbf{Sem}(R^*).$$

Since  $R^*$  is an over radical then  $R^* \leq P$  by Theorem 2.3. As  $R^*$  is a radical and  $P^\circ$  is a largest radical smaller than or equal to  $P$  then  $R^* \leq P^\circ$ .

- (4) is trivial.  $\square$

In other words, (1) and (2) of Proposition 3.4 state that the convolution and superposition procedures are isotone.

**3.2. The closure procedure.** The *closure procedure*  $P \mapsto \overline{P}$  makes a preradical  $P$  on a class  $\mathfrak{U}$  (containing  $\mathfrak{U}_n$ ) into a topological preradical  $\overline{P}$  on  $\mathfrak{U}_n$  by the rule

$$\overline{P}(A) = \overline{P(A)}$$

for every  $A \in \mathfrak{U}_n$ ;  $\overline{P}$  is called the *closure* of  $P$ . Dixon pointed out in [D3, Example 6.4] that if  $P$  is a radical then the closure  $\overline{P}$  may be not a topological radical, but always is a topological under radical. Recall that the restrictions of algebraic radicals from  $\mathfrak{U}_a$  to a class of normed algebras are algebraic under radicals. One can state Dixon's result [D3, Theorem 6.3] in the following stronger form.

**Theorem 3.5.** *The closure of the restriction of an algebraic under radical to a class of normed algebras is a strict topological under radical.*

*Proof.* We only prove the strictness for  $\overline{P}$ . Let  $f : A \rightarrow B$  be a continuous isomorphism of normed algebras. Then  $f(P(A)) = P(B)$  by definition whence  $f(\overline{P(A)})$  is dense in  $\overline{P(B)}$ . Therefore  $\overline{f(\overline{P(A)})} = \overline{P(B)}$ .  $\square$

Starting with a (restricted) algebraic radical  $P$ , one can take its closure and then apply the topological convolution procedure. The following theorem shows that the result will not change if we do the same with the algebraic convolution of  $P$ .

**Theorem 3.6.** *Let  $P$  be an algebraic under radical. Then  $\overline{P}^* = \overline{P^*}$  and  $\overline{P}^*$  is a strong topological radical.*

*Proof.* By Theorem 3.3,  $\mathbf{Sem}(P) = \mathbf{Sem}(P^*)$ . Then

$$\begin{aligned} \mathbf{Sem}(\overline{P}^*) &= \mathbf{Sem}(\overline{P}) = \mathfrak{U}_n \cap \mathbf{Sem}(P) = \mathfrak{U}_n \cap \mathbf{Sem}(P^*) \\ &= \mathbf{Sem}(\overline{P^*}) = \mathbf{Sem}(\overline{P^*}^*). \end{aligned}$$

As  $\overline{P}^*$  and  $\overline{P^*}^*$  are radicals,  $\overline{P}^* = \overline{P^*}^*$  by Corollary 2.4.

Let  $f : A \rightarrow B$  be a continuous surjective homomorphism of normed algebras, and let  $(P_\alpha)$  be the convolution series of  $\overline{P}$ . As  $P_1 = \overline{P}$  is strict by Theorem 3.5, it is strong. Let us prove by transfinite induction that every  $P_\alpha$  is strong.

Let  $\alpha$  be a limit ordinal, and let all  $P_{\alpha'}$  be strong for  $\alpha' < \alpha$ . Then

$$\begin{aligned} f(P_\alpha(A)) &= f\left(\overline{\bigcup_{\alpha' < \alpha} P_{\alpha'}(A)}\right) \subset \overline{f\left(\bigcup_{\alpha' < \alpha} P_{\alpha'}(A)\right)} \\ &= \overline{\bigcup_{\alpha' < \alpha} f(P_{\alpha'}(A))} \subset \overline{\bigcup_{\alpha' < \alpha} P_{\alpha'}(B)} = P_\alpha(B). \end{aligned}$$

The step  $\alpha \mapsto \alpha + 1$  of the induction is reduced to the consideration of  $\overline{P}$ : it suffices to check that

$$f(q_I^{-1}(\overline{P}(A/I))) \subset q_J^{-1}(\overline{P}(B/J)) \quad (3.7)$$

whenever  $f(I) \subset J$  where  $I = P_\alpha(A)$  and  $J = P_\alpha(B)$ . As  $\overline{P}$  is strong, (3.7) is easily checked. Indeed,  $f$  induces a continuous homomorphism  $f'$  from  $A/I$  onto  $B/J$  such that  $f' \circ q_I = q_J \circ f$ ; then

$$\begin{aligned} f(q_I^{-1}(\overline{P}(A/I))) &\subset q_J^{-1}(q_J(f(q_I^{-1}(\overline{P}(A/I))))) \\ &= q_J^{-1}(f'(\overline{P}(A/I))) \subset q_J^{-1}(\overline{P}(B/J)). \end{aligned}$$

As a consequence,  $\overline{P}^*$  is strong.  $\square$

We apply the above theorem to classical algebraic radicals related to the property of nilpotency. An algebra  $A$  is called *nilpotent* if there is an integer  $n > 0$  such that  $a_1 \cdots a_n = 0$  for every  $a_1, \dots, a_n \in A$ , *locally nilpotent* if every finite subset of  $A$  generates a nilpotent subalgebra, *nil* if every element of  $A$  is nilpotent. All these notions transfer to ideals.

An algebra  $A$  is called *semiprime* if it has no non-zero ideals with zero square. For semiprimeness of  $A$  it is sufficient to have no non-zero left or right ideals with zero square [BD, Lemma 30.4].

One can define the *Baer (prime or lower nil-) radical*  $\mathfrak{P}_\beta(A)$  of  $A$  as the smallest ideal  $I$  of  $A$  with semiprime quotient  $A/I$ ; the *Levitzki (or locally nilpotent) and Köte (or upper nil-) radicals*  $\mathfrak{P}_\lambda(A)$  and  $\mathfrak{P}_\kappa(A)$  are defined as the largest locally nilpotent and nil ideals of  $A$ , respectively. They all are hereditary radicals and

$$\mathfrak{P}_\beta < \mathfrak{P}_\lambda < \mathfrak{P}_\kappa. \quad (3.8)$$

Let  $\mathcal{P}_\beta = \overline{\mathfrak{P}_\beta}^*$ ,  $\mathcal{P}_\lambda = \overline{\mathfrak{P}_\lambda}^*$  and  $\mathcal{P}_\kappa = \overline{\mathfrak{P}_\kappa}^*$ . They are strong topological radicals by Theorems 3.5 and 3.1, and are called the *closed-Baer*, *closed-Levitzki* and *closed-Köte radical*, respectively.

Let  $A$  be an algebra in  $\mathfrak{U}_a$  and let  $\Sigma_\beta(A)$  be the sum of all nilpotent ideals of  $A$ . It is clear that  $\Sigma_\beta$  is an algebraic under radical on  $\mathfrak{U}_a$ ; by construction, the (algebraic) Baer radical  $\mathfrak{P}_\beta$  is equal to  $\Sigma_\beta^*$ .

Theorem 3.6 yields

**Corollary 3.7.**  $\mathcal{P}_\beta = \overline{\Sigma_\beta}^*$ .

**Theorem 3.8.**  $\mathcal{P}_\beta < \mathcal{P}_\lambda < \mathcal{P}_\kappa$  on  $\mathfrak{U}_n$  and  $\mathcal{P}_\beta = \mathcal{P}_\lambda = \mathcal{P}_\kappa$  on  $\mathfrak{U}_b$ .

*Proof.* As all procedures are isotone, the non-strict inequalities follows from (3.8). Also,  $\mathcal{P}_\beta < \mathcal{P}_\lambda$  follows from [D3, Proposition 3.3 and Corollary 9.4], and for the proof of  $\mathcal{P}_\lambda < \mathcal{P}_\kappa$  it is sufficient to point out a normed nil algebra with zero Levitzki radical; for instance, it was done in [T1] as some realization of Golod's solution of the Bernside problem for algebras.

By [D1],

$$\mathfrak{P}_\beta = \mathfrak{P}_\lambda = \mathfrak{P}_\kappa = \Sigma_\beta$$

on  $\mathfrak{U}_b$ . Therefore  $\mathcal{P}_\beta = \mathcal{P}_\lambda = \mathcal{P}_\kappa$  on  $\mathfrak{U}_b$ .  $\square$

The common radical on  $\mathfrak{U}_b$  is denoted by  $\mathcal{P}_{\text{nil}}$  and called the *closed-nil radical*. One can consider  $\mathcal{P}_\beta, \mathcal{P}_\lambda, \mathcal{P}_\kappa$  as topological extensions of  $\mathcal{P}_{\text{nil}}$  to normed algebras.

Let  $A$  be an algebra, and let  $\Sigma_{\text{hf}}(A)$  be the sum of all bifinite ideals of  $A$ . If  $A$  is normed, let  $\Sigma_{\text{hc}}(A)$  be the closure of sum of all bicomact ideals of  $A$ . It is easy to see that  $\Sigma_{\text{hf}}$  and  $\Sigma_{\text{hc}}$  are under radicals.

In the following theorem we apply the typical method of comparing radicals.

**Theorem 3.9.**  $\mathfrak{R}_{\text{hf}} = \Sigma_{\text{hf}}^*$  and  $\mathcal{R}_{\text{hf}} = \overline{\mathfrak{R}_{\text{hf}}}^* = \overline{\Sigma_{\text{hf}}}^* \leq \mathcal{R}_{\text{hc}} = \Sigma_{\text{hc}}^*$ .

*Proof.* Let  $B$  be an algebra. As every bifinite ideal is hypofinite,  $\Sigma_{\text{hf}}^*(B) \subset \mathfrak{R}_{\text{hf}}$ . Since  $\mathfrak{R}_{\text{hf}}$  is a radical,  $\Sigma_{\text{hf}}^* \leq \mathfrak{R}_{\text{hf}}$ . As every  $\Sigma_{\text{hf}}^*$ -semisimple algebra is  $\mathfrak{R}_{\text{hf}}$ -semisimple,  $\mathfrak{R}_{\text{hf}} \leq \Sigma_{\text{hf}}^*$ . One can show similarly that  $\mathcal{R}_{\text{hc}} = \Sigma_{\text{hc}}^*$ .

Let  $A$  be a normed algebra and  $J = \mathfrak{R}_{\text{hf}}(A)$ . Let  $(J_\alpha)_{\alpha \leq \gamma}$  be the increasing transfinite chain of ideals of  $A$  constructed in Theorem 2.20(5) with  $J_0 = 0$  and  $J_\gamma = J$ . As each  $J_{\alpha+1}/J_\alpha$  is bifinite,  $\overline{J_{\alpha+1}}/\overline{J_\alpha}$  is approximable, so  $\overline{J}$  is closed-hypofinite (see [ST6, Proposition 3.48]). As  $\mathcal{R}_{\text{hf}}(A)$  is the largest closed-hypofinite ideal of  $A$  then  $\overline{J} \subset \mathcal{R}_{\text{hf}}(A)$ . Therefore  $\overline{\mathfrak{R}_{\text{hf}}} \leq \mathcal{R}_{\text{hf}}$ , whence  $\overline{\mathfrak{R}_{\text{hf}}}^* \leq \mathcal{R}_{\text{hf}}$ . But every  $\overline{\mathfrak{R}_{\text{hf}}}^*$ -semisimple algebra has no non-zero finite rank elements and is therefore  $\mathcal{R}_{\text{hf}}$ -semisimple, whence  $\mathcal{R}_{\text{hf}} \leq \overline{\mathfrak{R}_{\text{hf}}}^*$  by Theorem 2.3.

By Theorem 3.6,  $\mathcal{R}_{\text{hf}} = \overline{\Sigma_{\text{hf}}}^*$ . As  $\overline{\Sigma_{\text{hf}}} \leq \Sigma_{\text{hc}}$  then  $\mathcal{R}_{\text{hf}} = \overline{\Sigma_{\text{hf}}}^* \leq \Sigma_{\text{hc}}^* = \mathcal{R}_{\text{hc}}$  by Theorem 3.4.  $\square$

**3.3. The regularization procedure.** Let  $P$  be a hereditary topological radical defined on some class  $\mathfrak{U}$  of normed algebras such that  $\mathfrak{U}_b \subset \mathfrak{U}$ . By [ST5, Theorem 2.21], the map  $P^r$  defined by

$$P^r(A) = A \cap P(\widehat{A}) \quad (3.9)$$

for each  $A \in \mathfrak{U}_n$ , where  $\widehat{A}$  is a completion of  $A$ , is a hereditary topological radical on  $\mathfrak{U}_n$ . The map  $P \mapsto P^r$  is called the *regularization procedure*; it is clear that

$$P^{rr} = P^r.$$

A hereditary topological radical  $P$  on  $\mathfrak{U}_n$  is called *regular* if  $P = P^r$ . One of the important reasons for the consideration of over radicals is the fact [ST5] that preradicals obtained from topological radicals by the regular procedure are over radicals.

**Theorem 3.10.** *The radicals  $\mathcal{R}_{cq}$  and  $\mathcal{R}_t$  are regular.*

*Proof.* Let  $A$  be a normed algebra. By definition,  $A \cap \mathcal{R}_{cq}(\widehat{A}) \subset \mathcal{R}_{cq}(A)$ . On the other hand, the completion of a compactly quasinilpotent algebra is again compactly quasinilpotent [ST5, Lemma 4.11]. So  $\overline{\mathcal{R}_{cq}(A)}^{(\widehat{A})}$  is compactly quasinilpotent, and as it is an ideal of  $\widehat{A}$  then  $\overline{\mathcal{R}_{cq}(A)}^{(\widehat{A})} \subset \mathcal{R}_{cq}(\widehat{A})$ . But  $A \cap \overline{\mathcal{R}_{cq}(A)}^{(\widehat{A})} = \mathcal{R}_{cq}(A)$  because  $\mathcal{R}_{cq}(A)$  is a closed ideal of  $A$ . Thus  $\mathcal{R}_{cq}$  is regular.

The proof for  $\mathcal{R}_t$  is similar. For the other proof we refer to Proposition 5.17.  $\square$

**Theorem 3.11.** *The radicals  $\mathcal{R}_\beta$  and  $\mathcal{R}_{hf}$  are not regular.*

*Proof.* Let  $B$  be the Banach space  $L^1[0, 1]$  considered as a Banach algebra with multiplication  $(f * g)(t) = \int_0^t f(s)g(t-s)ds$ . It can be shown that nilpotent elements are dense in  $B$  (and therefore all elements of  $B$  are quasinilpotent since  $B$  is commutative): indeed, it follows from the Titchmarsh convolution theorem that a function  $f \in B$  is nilpotent if and only if it vanishes a.e. on some interval  $(0, a)$ . Therefore  $\mathcal{R}_\beta(B) = B$ . Moreover,  $f$  is an element of finite rank if it is nilpotent of index 2 (equivalently if  $f(t) = 0$  for  $t \in (0, 1/2)$ ), so  $\mathcal{R}_{hf}(B) \neq 0$ . In fact,  $\mathcal{R}_{hf}(B) = B$ .

Let  $A$  be the subalgebra of  $B$  that consists of polynomials. It is dense in  $B$  so  $B$  can be considered as the completion of  $A$ . Clearly  $A$  has no nilpotent elements (a non-zero polynomial cannot vanish on an interval), so  $\mathcal{R}_\beta(A) = 0$ . As a quasinilpotent finite rank element is nilpotent,  $A$  has no finite rank elements and  $\mathcal{R}_{hf}(A) = 0$ .  $\square$

**Problem 3.12.** *Is the hypocompact radical  $\mathcal{R}_{hc}$  regular?*

It is clear that the regular Jacobson radical  $\text{Rad}^r$  defined by (2.4) is a regular radical. Note that

$$\mathcal{R}_{cq} \leq \mathcal{R}_t \leq \text{Rad}^r < \text{rad}_n < \text{rad}_b; \quad (3.10)$$

also  $\text{rad} = \text{rad}_n$  on  $\mathfrak{U}_q$  and  $\text{rad} = \text{rad}_b$  on  $\mathfrak{U}_{qb}$  [ST5, Theorem 2.18].

Consider the *closed-Jacobson radical*  $\overline{\text{rad}}^*$ ; then  $\overline{\text{rad}}^* = \text{Rad}$  on  $\mathfrak{U}_b$ . Define also the map  $\Pi_{pc}$  on normed algebras by

$$\Pi_{pc}(A) = \cap \{I \in \text{Prim}(A^1) : I \text{ is closed in } A\}. \quad (3.11)$$

**Remark 3.13.** *The map  $A \mapsto \{I \in \text{Prim}(A^1) : I \text{ is closed in } A\}$  is not a primitive map because, for an ideal  $J$  of a normed algebra  $A$ , there can exist a non-closed primitive ideal  $I$  of  $A$  such that  $I \cap J$  is a closed primitive ideal of  $J$ .*

**Lemma 3.14.**  *$\Pi_{pc}$  is a topological over radical on normed algebras, and  $\overline{\text{rad}} \leq \Pi_{pc}$ .*

*Proof.* Let  $A$  be a normed algebra, and let  $I$  be an ideal of  $A$ . If  $J \in \text{Prim}(A^1)$  is closed then  $I \cap J \in \text{Prim}(I^1)$  is closed in  $I$ , whence  $\Pi_{\text{pc}}(I) \subset \Pi_{\text{pc}}(A)$  is an ideal of  $A$  by (1.9).

Let  $B$  be a normed algebra, and let  $f : A \rightarrow B$  be a morphism. Let  $\pi \in \text{Irr}(B^1)$  and  $K := \ker \pi$  is closed. Then  $\pi \circ f \in \text{Irr}(A^1)$  and  $\ker(\pi \circ f) = f^{-1}(K)$  is closed.

Let  $\mathcal{F}_A$  be the set of all closed primitive ideals of  $A^1$ . Assume that  $I \subset \Pi_{\text{pc}}(A)$  is closed. Then  $I \subset J$  for every  $J \in \mathcal{F}_A$  and

$$\begin{aligned} \Pi_{\text{pc}}(A)/I &= \cap \{J : J \in \mathcal{F}_A\}/I = \cap \{J/I : J \in \mathcal{F}_A\} \\ &= \cap \{J/I : J/I \in \mathcal{F}_{A/I}\} = \Pi_{\text{pc}}(A/I) \end{aligned}$$

by (1.9) and (1.10). Hence  $\Pi_{\text{pc}}$  is an over radical.

Furthermore,  $\overline{\text{rad}}(A) = \overline{\cap_{I \in \text{Prim}(A^1)} I} \subset \Pi_{\text{pc}}(A)$  in virtue of (3.11).  $\square$

As a consequence,  $\Pi_{\text{pc}}^\circ$  is a topological radical which also extends  $\text{Rad}$  to normed algebras; it is called the *primitively closed Jacobson radical*.

**Theorem 3.15.**  $\overline{\text{rad}}^* < \Pi_{\text{pc}}^\circ \leq \text{rad}_n$ ;  $\overline{\text{rad}}^*$  and  $\text{Rad}^r$  are not comparable.

*Proof.* Indeed, it is clear that  $\overline{\text{rad}}(A) \subset \Pi_{\text{pc}}(A) \subset \text{rad}_n(A)$  for every normed algebra  $A$ . Then  $\overline{\text{rad}}^* \leq \Pi_{\text{pc}}^\circ \leq \text{rad}_n$  by Theorems 2.3 and 3.4.

Let  $a$  be a quasinilpotent element of a Banach algebra and  $a^n \neq 0$  for every  $n > 0$ . Let  $A$  be the algebra generated by  $a$ , and  $B = \overline{A}$ . Then  $\text{rad}(A) = 0$ , but  $\text{rad}(B) = B$ . Hence  $\overline{\text{rad}}^*(A) = \overline{\text{rad}}(A) = 0$ , but  $\text{Rad}^r(A) = A \cap \text{rad}(B) = A$ .

Also,  $A$  has no closed primitive ideals in  $\text{Prim}(A^1)$  besides of  $A$ . Indeed, a primitive ideal  $I$  is the kernel of multiplicative functional  $f$  on  $A^1$ . If  $I \neq A$  is closed then  $f(a) \neq 0$ ,  $f$  is continuous and  $f(a)$  in the spectrum  $\sigma_B(a)$ , a contradiction. Therefore  $\Pi_{\text{pc}}^\circ(A) = \Pi_{\text{pc}}(A) = A$  and  $\overline{\text{rad}}^* < \Pi_{\text{pc}}^\circ$ .

By [D3, Example 9.3], there are Banach algebras  $C$ ,  $D$  and an injective embedding  $\phi : C \rightarrow D$  with the dense image such that  $\text{rad}(C) = C$  and  $\text{rad}(D) = 0$ . Let  $E = \phi(C)$ . As  $\phi$  and  $\phi^{-1}$  are algebraic morphisms for appropriate algebras then  $\text{rad}(E) = E$  and  $\overline{\text{rad}}^*(E) = E$ , but  $\text{Rad}^r(E) = E \cap \text{rad}(D) = 0$ . So the observations with algebras  $A$  and  $E$  show that  $\overline{\text{rad}}^*$  and  $\text{Rad}^r$  are not comparable.  $\square$

As we will see in Theorem 4.31, the closed-Jacobson radical  $\overline{\text{rad}}^*$  is hereditary.

#### 4. OPERATIONS WITH RADICALS

Operations are multiplace procedures that act on a class of ideal maps.

**4.1. Supremum and infimum.** Let  $\mathcal{F}$  be a non-empty family (set, class) of radicals. Then the maps  $H_{\mathcal{F}}$  and  $B_{\mathcal{F}}$  defined by

$$\begin{aligned} H_{\mathcal{F}}(A) &= \sum_{P \in \mathcal{F}} P(A) & [H_{\mathcal{F}}(A) &= \overline{\sum_{P \in \mathcal{F}} P(A)}] \\ B_{\mathcal{F}}(A) &= \cap_{P \in \mathcal{F}} P(A) \end{aligned}$$

are the under and over radicals, respectively (see [ST7, Lemma 3.2]). We extend the statement to under/over radicals as follows.

**Theorem 4.1.** (1) If  $\mathcal{F}_u$  is a family of under radicals then  $H_{\mathcal{F}_u}$  is an under radical, and

$$\text{Sem}(H_{\mathcal{F}_u}) = \cap_{P \in \mathcal{F}_u} \text{Sem}(P).$$

(2) If  $\mathcal{F}_o$  is a family of over radicals then  $B_{\mathcal{F}_o}$  is an over radical, and

$$\text{Rad}(B_{\mathcal{F}_o}) = \cap_{P \in \mathcal{F}_o} \text{Rad}(P).$$

*Proof.* The equalities for radical and semisimple classes follow from the definitions of  $H_{\mathcal{F}_u}$  and  $B_{\mathcal{F}_o}$ . It is clear that  $H_{\mathcal{F}_u}$  and  $B_{\mathcal{F}_o}$  satisfy Axioms 1 and 4.

Let  $A$  be an algebra. If  $P \in \mathcal{F}_u$  then  $P(A)$  is an ideal in  $H_{\mathcal{F}_u}(A)$  and therefore

$$P(A) = P(P(A)) \subset P(H_{\mathcal{F}_u}(A)) \subset H_{\mathcal{F}_u}(H_{\mathcal{F}_u}(A))$$

whence  $H_{\mathcal{F}_u}(A) \subset H_{\mathcal{F}_u}(H_{\mathcal{F}_u}(A))$ . This proves that  $H_{\mathcal{F}_u}$  is an under radical.

Let  $I = B_{\mathcal{F}_o}(A)$  and  $J = q_I^{-1}(B_{\mathcal{F}_o}(A/I))$  where  $q_I : A \rightarrow A/I$  is the standard quotient map. Let  $R \in \mathcal{F}_o$ ; then  $R(A) \subset I$  and there is an [open continuous] epimorphism  $p : A/I \rightarrow A/R(A)$  such that  $q = p \circ q_I$  where  $q : A \rightarrow A/R(A)$  is the standard quotient map. Then

$$q(J) = (p \circ q_I)(J) = p(B_{\mathcal{F}_o}(A/I)) \subset p(R(A/I)) \subset R(A/R(A)) = 0,$$

whence  $J \subset R(A)$  for every  $R \in \mathcal{F}_o$ . Then  $J \subset I$  and  $B_{\mathcal{F}_o}(A/I) = q_I(J) = 0$ . This means that  $B_{\mathcal{F}_o}$  is an over radical.  $\square$

From the viewpoint of the order,  $H_{\mathcal{F}_u}$  is the supremum of  $\mathcal{F}_u$  in the class of under radicals and  $B_{\mathcal{F}_o}$  is the infimum of  $\mathcal{F}_o$  in the class of over radicals.

In general, for a family  $\mathcal{F}$  of preradicals satisfying Axiom 4, let us define the ideal maps  $\vee \mathcal{F}$  and  $\wedge \mathcal{F}$  by

$$\vee \mathcal{F} = H_{\mathcal{F}}^* \quad \text{and} \quad \wedge \mathcal{F} = B_{\mathcal{F}}^o,$$

respectively; if  $\mathcal{F}$  is  $\{P_1, P_2\}$  then instead we write  $P_1 \vee P_2$  or  $P_1 \wedge P_2$ , respectively.

**Remark 4.2.** It is easy to check (arguing as in Theorems 4.1 and 3.1) that, for a family  $\mathcal{F}$  of preradicals satisfying Axiom 4,  $\vee \mathcal{F}$  determines the smallest over radical larger than or equal to each  $R \in \mathcal{F}$  and  $\wedge \mathcal{F}$  determines the largest under radical smaller than or equal to each  $P \in \mathcal{F}$ .

**Corollary 4.3.** The operations  $\vee$  and  $\wedge$  produce supremum and infimum of a family of radicals in the class of radicals.

*Proof.* Let  $\mathcal{F}$  be a family of radicals. Let  $P$  be a radical such that  $R \leq P$  for every  $R \in \mathcal{F}$ . Then  $H_{\mathcal{F}} \leq P$ , whence  $\vee \mathcal{F} \leq P$  by Theorem 4.1 and Proposition 3.4. As  $R \leq \vee \mathcal{F}$  for every  $R \in \mathcal{F}$ , we conclude that  $\vee \mathcal{F}$  is the supremum of  $\mathcal{F}$ .

Similarly,  $\wedge \mathcal{F}$  is the infimum of  $\mathcal{F}$ .  $\square$

Taking into account Theorems 2.3 and 3.3, one can reformulate Theorem 3.1 as follows.

**Theorem 4.4.** Let  $R$  be an under radical, and let  $P$  be an over radical. Then

- (1)  $R^*$  is the only radical  $S$  such that  $\mathbf{Sem}(S) = \mathbf{Sem}(R)$ ;
- (2)  $P^o$  is the only radical  $T$  such that  $\mathbf{Rad}(T) = \mathbf{Rad}(P)$ .

In assumptions of Theorem 4.4, we also recall that

$$\mathbf{Rad}(R) \subset \mathbf{Rad}(R^*) \quad \text{and} \quad \mathbf{Sem}(P) \subset \mathbf{Sem}(P^o).$$

**Corollary 4.5.** Let  $V$  be a class of algebras. Then there exist the largest radical  $S$  such that  $\mathbf{Sem}(S) \supset V$ , and the smallest radical  $T$  such that  $\mathbf{Rad}(T) \supset V$ .

- (1)  $V = \mathbf{Sem}(S)$  if and only if there is a family  $\mathcal{F}_1$  of under radicals such that  $\mathbf{Sem}(H_{\mathcal{F}_1}) = V$ ;
- (2)  $V = \mathbf{Rad}(T)$  if and only if there is a family  $\mathcal{F}_2$  of over radicals such that  $\mathbf{Rad}(B_{\mathcal{F}_2}) = V$ .

*Proof.* Let  $\mathcal{F}_1$  be the family of all under radicals  $P$  such that  $\mathbf{Sem}(P) \supset V$ , and let  $\mathcal{F}_2$  be the family of all over radicals  $P$  such that  $\mathbf{Rad}(P) \supset V$ . By Theorems 4.4 and 4.1, there exist radicals  $S$  and  $T$  such that  $\mathbf{Sem}(S) = \cap_{P \in \mathcal{F}_1} \mathbf{Sem}(P) \supset V$  and  $\mathbf{Rad}(T) = \cap_{P \in \mathcal{F}_2} \mathbf{Rad}(P) \supset V$ .  $\square$

For instance, let  $W_\rho$  be the class of all normed algebras on which the usual spectral radius  $a \mapsto \rho(a)$  is continuous; this class contains all commutative normed algebras and Banach algebras whose elements have at most countable spectrum. Then there exists a largest topological radical  $S_\rho$  such that  $W_\rho \subset \mathbf{Sem}(S_\rho)$ , but it is a problem whether there is a largest topological radical  $T$  for which every  $T$ -radical normed algebra has the property of spectral radius continuity, i.e. lies in  $W_\rho$ .

Let  $V$  be a class of algebras. For being a semisimple or radical class it is necessary that  $V$  is stable under extensions.

**Theorem 4.6.** *Let  $A$  be an algebra, and let  $I$  be a [closed] ideal of  $A$ . Then*

- (1) *If  $P$  is an under radical and  $I, A/I$  are  $P$ -semisimple then  $A$  is  $P$ -semisimple;*
- (2) *If  $P$  is an over radical and  $I, A/I$  are  $P$ -radical then  $A$  is  $P$ -radical.*

*Proof.* (1) As  $q_I(P(A)) \subset P(A/I) = 0$  then  $P(A) \subset I$ , whence  $P(A)$  is an ideal of  $I$  and  $P(A) = P(P(A)) \subset P(I) = 0$ .

(2) As  $I \subset P(A)$  then there is a morphism  $p : A/I \rightarrow A/P(A)$  such that  $q_{P(A)} = p \circ q_I$  where  $q_{P(A)}$  and  $q_I$  are standard quotient maps. Then

$$A/P(A) = p(A/I) = p(P(A/I)) \subset P(A/P(A)) = 0,$$

whence  $A = P(A)$ .  $\square$

In other words, the class  $\mathbf{Rad}(P)$  (respectively,  $\mathbf{Sem}(P)$ ) is stable under extensions if  $P$  is an over radical (respectively, under radical). We will need also partially converse statements.

**Lemma 4.7.** *Let  $R$  be an under radical, and let  $P$  be an over radical. Then*

- (1) *If  $\mathbf{Rad}(R)$  is stable under extensions then  $R$  is a radical;*
- (2) *If  $\mathbf{Sem}(P)$  is stable under extensions then  $P$  is a radical.*

*Proof.* (1) Let  $J = q_{R(A)}^{-1}(R(A/R(A)))$ . Then the ideal  $J$  is an extension of  $R(A)$  by  $R(A/R(A))$ : both algebras are  $R$ -radical because  $R$  is an under radical. By our assumption,  $J$  is a  $R$ -radical ideal of  $A$ . Therefore  $J \subset R(A)$  which means that  $R(A/R(A)) = 0$ , i.e.,  $R$  is a radical.

(2) Let  $J = P(P(A))$ . Then the algebra  $A/J$  is an extension of  $P(A)/J$  by  $(A/J)/(P(A)/J) \cong A/P(A)$ . Both algebras are  $P$ -semisimple because  $P$  is an over radical. By our assumption,  $A/J$  is  $P$ -semisimple. Then

$$P(A)/J = q_J(P(A)) \subset P(A/J) = 0,$$

whence  $J = P(A)$ , i.e.,  $P$  is a radical.  $\square$

We apply this result to preradicals  $\overline{\Pi_{\Omega_n}}$  and  $\overline{\Pi_{\Omega_b}}$ , generated by the primitive maps  $\Omega_n := \text{Prim} \setminus \text{Prim}_n$  and  $\Omega_b := \text{Prim} \setminus \text{Prim}_b$  on  $\mathfrak{U}_n$  (see Example 2.14).

**Theorem 4.8.**  *$\overline{\Pi_{\Omega_n}}$  and  $\overline{\Pi_{\Omega_b}}$  are radicals on  $\mathfrak{U}_n$ ; for every normed algebra  $A$ ,  $\overline{\Pi_{\Omega_n}}(A)$  is the largest  $Q$ -ideal of  $A$  and  $\overline{\Pi_{\Omega_b}}(A)$  is the largest  $Q_b$ -ideal of  $A$ .*

*Proof.* As  $\Pi_{\Omega_n}$  is a hereditary preradical with topological morphisms then  $\overline{\Pi_{\Omega_n}}$  is an under radical by Theorem 3.5. Since  $\Pi_{\Omega_n}(A)$  is a  $Q$ -algebra,  $\overline{\Pi_{\Omega_n}}(A) := \overline{\Pi_{\Omega_n}(A)}$  is a  $Q$ -algebra. Taking into account that every  $Q$ -algebra is a  $\Pi_{\Omega_n}$ -radical algebra by definition, and the class of all  $Q$ -algebras is stable under extensions (see for instance [ST5, Theorem 2.5]), we conclude that  $\overline{\Pi_{\Omega_n}}$  is a radical, by Lemma 4.7. It is clear that  $\overline{\Pi_{\Omega_n}}(A)$  is the largest  $Q$ -ideal of  $A$ .

A similar argument gives the result for  $\overline{\Pi_{\Omega_b}}$ .  $\square$

**4.2. Superposition and convolution operations.** The *superposition* of preradicals  $P$  and  $R$  satisfying Axiom 4 is defined by the usual rule:

$$(P \circ R)(A) = P(R(A))$$

for each algebra  $A$ . This map is a preradical satisfying Axiom 4 by Proposition 3.2. The superposition operation is clearly associative.

Let  $A$  be an algebra, and let  $P$  and  $R$  be ideal maps. For a [closed] ideal  $I$  of  $A$ , define the ideal  $P * I$  of  $A$  by

$$P * I := q_I^{-1}(P(A/I)),$$

where  $q_I : A \rightarrow A/I$  is the standard quotient map; in particular  $P * 0 = P(A)$ ,  $P * P(A) = q_{P(A)}^{-1}(P(A/P(A)))$  and  $P * A = A$ . Sometimes we will write  $(P * I; A)$  instead of  $P * I$  to avoid a misunderstanding.

Define the *convolution*  $P * R$  of ideal maps  $P, R$  by

$$(P * R)(A) := P * R(A) = q_{R(A)}^{-1}(P(A/R(A))). \quad (4.1)$$

Proposition 3.2 shows that if  $P$  and  $R$  are preradicals satisfying Axiom 4 then  $P * R$  is a preradical satisfying Axiom 4.

The operations of superposition and convolution are related by the dual procedure  $P \mapsto P^\dagger$ , where  $P^\dagger$  is a *quotient-valued map* corresponding to  $P$ :

$$P^\dagger(A) = A/P(A)$$

for every algebra  $A$ . This relation reflected in Proposition 4.9 below reminds the Fourier transform which relates convolution and product of functions. Rewrite (4.1) as

$$(P * R)(A) = q_{R(A)}^{-1}(P(R^\dagger(A)))$$

for every  $A$ .

Let  $S$  be a quotient-valued map, i.e. let  $S$  send every algebra  $A$  to  $A/I$  for some ideal  $I$ . The dependence of  $I$  on  $A$  can be written as  $I = S^\dagger(A)$ , so we have an ideal-valued map  $S^\dagger$  such that

$$S(A) = A/S^\dagger(A) = q_{S^\dagger(A)}(A)$$

for every algebra  $A$ . If  $T, S$  are quotient-valued maps then their *superposition*  $T \circ S$  is defined by

$$(T \circ S)(A) := A/q_{S^\dagger(A)}^{-1}(T^\dagger(S(A))) \cong S(A)/T^\dagger(S(A)) = T(S(A))$$

for every algebra  $A$ .

**Proposition 4.9.** *Let  $P, R$  be ideal-valued maps. Then*

$$(R * P)^\dagger = R^\dagger \circ P^\dagger.$$

*Proof.* If  $J$  is a [closed] ideal of  $A$  and  $I$  is a [closed] ideal of  $A/J$  then one identifies in a standard way the algebras  $(A/J)/I$  and  $A/q_J^{-1}(I)$ . Therefore

$$\begin{aligned} R^\dagger \circ P^\dagger(A) &= (A/P(A))/R(A/P(A)) = A/q_{P(A)}^{-1}(R(A/P(A))) \\ &= A/(R * P)(A) = (R * P)^\dagger(A). \end{aligned}$$

□

**Lemma 4.10.** *Let  $P$  and  $R$  be preradicals. Then*

- (1)  $P * R$  is a preradical;
- (2) The convolution operation is associative.



*Proof.* (1) Let  $f : A \rightarrow B$  be a morphism. Then  $f(P(A)) \subset P(B)$ , so there is a morphism  $g : q_{P(A)}(A) \rightarrow q_{P(B)}(B)$  such that  $g \circ q_{P(A)} = q_{P(B)} \circ f$ ; then  $g(R(q_{P(A)}(A))) \subset R(q_{P(B)}(B))$ . Let  $a \in (P*R)(A)$ ; then

$$(g \circ q_{P(A)})(a) \in R(q_{P(B)}(B)),$$

whence  $(q_{P(B)} \circ f)(a) \in R(q_{P(B)}(B))$  and  $f(a) \in (P*R)(B)$ .

(2) Let  $K$  be a preradical. We have that

$$(R*K)*P(A) = q_{I_1}^{-1}(R*K(A/I_1)) = q_{I_1}^{-1}(q_{I_2}^{-1}(R((A/I_1)/I_2)))$$

where  $I_1 = P(A)$ ,  $I_2 = K(A/I_1)$ . Also,  $R*(K*P)(A) = q_{I_3}^{-1}(R(A/I_3))$  where

$$I_3 = K*P(A) = q_{I_1}^{-1}(K(A/I_1)) = q_{I_1}^{-1}(I_2).$$

It remains to note that  $q_{I_3} = q_{I_2} \circ q_{I_1}$  and  $(A/I_1)/I_2 \cong A/q_{I_1}^{-1}(I_2) = A/I_3$ .  $\square$

The following result is a direct consequence of Proposition 3.2.

**Corollary 4.11.** (1) *If  $P$  and  $R$  are under radicals then  $P*R$  is an under radical.*

(2) *If  $P$  and  $R$  are over radicals then  $P \circ R$  is an over radical.*

**Theorem 4.12.** *If  $P$  and  $R$  are hereditary under radicals defined on a universal class  $\mathfrak{U}$ , and  $P$  is pliant then  $P*R$  is a hereditary under radical.*

*Proof.* Let  $A$  be an algebra,  $J$  an ideal of  $A$ . As  $P$  and  $R$  are under radicals then  $P*R$  is also an under radical by Corollary 4.11. Therefore

$$(P*R)(J) \subset J \cap (P*R)(A).$$

We have to prove the converse inclusion.

Since  $R(J) = J \cap R(A)$ , setting  $f(x/R(J)) = q_{R(A)}(x)$  we obtain an isomorphism  $f$  of the algebra  $J/R(J)$  onto the ideal  $q_{R(A)}(J)$  of  $A/R(A)$ . Since  $P$  is pliant,  $f(P(J/R(J))) = P(q_{R(A)}(J))$ .

If  $x \in J \cap (P*R)(A)$  then

$$q_{R(A)}(x) \in P(A/R(A)) \cap q_{R(A)}(J) = P(q_{R(A)}(J)) = f(P(J/R(J))).$$

Thus  $f(x/R(J)) \in f(P(J/R(J)))$ , whence  $x/R(J) \in P(J/R(J))$  and  $x \in (P*R)(J)$ . Therefore  $J \cap (P*R)(A) \subset (P*R)(J)$  and the proof is complete.  $\square$

**Remark 4.13.** *The proof of Theorem 4.12 shows that it remains true if  $P, Q$  are topological radicals on Banach algebras and  $P$  satisfies the condition of Banach heredity (2.2).*

For brevity, let

$$P + R := H_{\{P, R\}} \text{ and } P \cdot R := B_{\{P, R\}}$$

for preradicals  $P$  and  $R$ .

**Lemma 4.14.** (1) *Let  $P$  and  $R$  be under radicals. Then*

$$P + R \leq P*R \leq (P + R) * (P + R).$$

(2) *Let  $P$  and  $R$  be over radicals. Then*

$$(P \cdot R) \circ (P \cdot R) \leq P \circ R \leq P \cdot R.$$

*Proof.* (1) Let  $A$  be an arbitrary algebra,  $I = P(A)$ ,  $J = (P + R)(A)$ , and let  $q_I : A \rightarrow A/I$  and  $q_J : A \rightarrow A/J$  be standard quotient maps. As

$$q_I(R(A)) \subset R(A/I) = q_I((P*R)(A))$$

then  $P + R \leq P * R$ . As  $I \subset J$ , there is a morphism  $q : A/I \rightarrow A/J$  such that  $q \circ q_I = q_J$ . Therefore

$$q_J((P * R)(A)) = q(R(A/I)) \subset R(A/J) \subset (P + R)(A/J)$$

whence  $(P * R)(A) \subset ((P + R) * (P + R))(A)$ .

(2) Clearly

$$(P \circ R)(A) = P(R(A)) \subset P(A) \cap R(A) = (P \cdot R)(A).$$

Let  $I = P(A) \cap R(A)$ . As  $I$  is an ideal of  $R(A)$  then

$$(P \cdot R)(I) \subset P(I) \subset P(R(A)).$$

□

**Corollary 4.15.** (1) *Let  $P$  and  $R$  be under radicals. Then*

$$(P * R)^* = (R * P)^* = P \vee R = P^* \vee R^*.$$

(2) *Let  $P$  and  $R$  be over radicals. Then*

$$(P \circ R)^\circ = (R \circ P)^\circ = P \wedge R = P^\circ \wedge R^\circ.$$

*Proof.* (1) follows from Lemma 4.14 if one takes into account that

$$((P + R) * (P + R))^* = P \vee R$$

by associativity of the convolution.

(2) follows from Lemma 4.14 if one takes into account that

$$((P \cdot R) \circ (P \cdot R))^\circ = P \wedge R$$

by associativity of the superposition. □

This corollary can be easily extended to finite and even transfinite “products” of corresponding maps.

It is straightforward that if  $P$  and  $R$  are hereditary radicals then  $P \cdot R$  is a hereditary radical. As a consequence of this, Lemma 4.14(2) and Corollary 4.15, we obtain the following

**Corollary 4.16.** *Let  $P$  and  $R$  be the hereditary radicals. Then  $P \circ R$  is a hereditary radical and*

$$P \circ R = R \circ P = P \wedge R = R \cdot P.$$

**4.3. Transfinite chains of ideals and radicals.** We start with a simple observation on the quotient algebras related to preradicals.

**Proposition 4.17.** *Let  $A$  be an algebra, and let  $I$  be an ideal of  $A$ . Then*

- (1) *If  $P$  is an over radical and*
  - (a)  *$P(A) \subset I$  then  $I/P(A)$  is  $P$ -semisimple;*
  - (b)  *$I \subset P(A)$  [for closed  $I$ ] then  $P(A/I) = P(A)/I$ .*
- (2) *If  $P$  is a hereditary preradical and  $I \subset P(A)$  [for closed  $I$ ] then  $P(A)/I$  is  $P$ -radical.*

*Proof.* (1a)  $P(I/P(A)) \subset P(A/P(A)) = 0$ .

(1b) There is a morphism  $p : A/I \rightarrow A/P(A)$  such that  $q_{P(A)} = p \circ q_I$  where  $q_{P(A)}$ ,  $q_I$  are the corresponding standard quotient maps. Then

$$p(P(A/I)) \subset P(A/P(A)) = 0$$

whence  $P(A/I) \subset \ker p = P(A)/I = q_I(P(A)) \subset P(A/I)$ .

(2) Indeed,  $P(A)/I = q_I(P(A)) \subset P(A/I)$ . Hence, by heredity of  $P$ ,

$$P(P(A)/I) = (P(A)/I) \cap P(A/I) = P(A)/I.$$

□

Let  $P$  be a preradical, and let  $A$  be an algebra. A [closed] ideal  $I$  of  $A$  is called *P-absorbing* if  $A/I$  is  $P$ -semisimple.

**Theorem 4.18.** *Let  $A$  be an algebra,  $P$  an over radical,  $R$  an under radical. Then*

- (1) *[The closure of] the sum of any family of  $R$ -radical ideals is  $R$ -radical;*
- (2) *If  $(J_\alpha)_{\alpha \leq \gamma}$  is an increasing transfinite chain of ideals of  $A$  with  $J_0 = 0$  and all gap-quotients are  $R$ -radical then  $J_\gamma$  is  $R$ -radical;*
- (3) *The intersection of any number of  $P$ -absorbing ideals is  $P$ -absorbing;*
- (4) *If  $(I_\alpha)_{\alpha \leq \delta}$  is a decreasing transfinite chain  $(I_\alpha)_{\alpha \leq \gamma}$  of ideals of  $A$  with  $I_0 = A$  and all  $I_\alpha/I_{\alpha+1}$  are  $P$ -semisimple then  $I_\delta$  is  $P$ -absorbing.*

*Proof.* (1) Let  $I$  be the [closure of the] sum of a family  $(J_\alpha)_{\alpha \in \Lambda}$  of  $P$ -radical ideals of  $A$ . Then  $J_\alpha = R(J_\alpha) \subset R(I) \subset I$  for every  $\alpha$ , whence  $I = R(I)$ .

(2) follows by transfinite induction. The step  $\alpha \mapsto \alpha + 1$  follows from Theorem 4.6, and the case of limit ordinals follows from (1).

(3) Let  $I = \cap_{\alpha \in \Lambda} J_\alpha$  be the intersection of  $P$ -absorbing ideals of  $A$ . As  $I \subset J_\alpha$  for every  $\alpha$ , then, for the standard morphism  $q_\alpha : A/I \rightarrow A/J_\alpha$ , we have  $q_\alpha(P(A/I)) \subset P(A/J_\alpha) = 0$ . Therefore  $P(A/I) \subset \cap_{\alpha \in \Lambda} \ker q_\alpha = \cap_{\alpha \in \Lambda} J_\alpha/I = (\cap_{\alpha \in \Lambda} J_\alpha)/I = 0$ .

(4) follows by transfinite induction. The step  $\alpha \mapsto \alpha + 1$  follows from Theorem 4.6, and the case of limit ordinals follows from (3).  $\square$

**Theorem 4.19.** *Let  $\mathcal{F}_u$  be a family of under radicals, let  $\mathcal{F}_o$  be a family of over radicals, and let  $A$  be a [normed] algebra. Then*

- (1) *There is an increasing transfinite chain  $(I_\alpha)_{\alpha \leq \gamma}$  of [closed] ideals of  $A$  such that  $I_0 = 0$ ,  $I_\gamma = (\vee \mathcal{F}_u)(A)$  and every gap-quotient  $I_{\alpha+1}/I_\alpha$  of the chain is  $P$ -radical and is equal to  $P(A/I_\alpha)$  for some  $P \in \mathcal{F}_u$ ;*
- (2) *There is an increasing transfinite chain  $(J_\alpha)_{\alpha \leq \delta}$  of [closed] ideals of  $A$  such that  $J_0 = A$ ,  $J_\delta = (\wedge \mathcal{F}_o)(A)$  and every gap-quotient of the chain is  $P$ -semisimple for some  $P \in \mathcal{F}_o$ .*

*Proof.* (1) Let  $I = (\vee \mathcal{F}_u)(A)$ . Arguing by transfinite induction, one can assume that we already have the required chain  $(I_{\alpha'})_{\alpha' \leq \alpha}$  but with the only distinction  $I_\alpha \subset I$ . If  $I_\alpha \neq I$  then  $I/I_\alpha$  is a non-zero  $\vee \mathcal{F}_u$ -radical algebra by Proposition 4.17(1b), because  $\vee \mathcal{F}_u$  is a radical. Therefore  $H_{\mathcal{F}_u}(I/I_\alpha) \neq 0$ . Then  $H_{\mathcal{F}_u}(A/I_\alpha) \neq 0$ , and there is some  $P \in \mathcal{F}_u$  such that  $P(A/I_\alpha) \neq 0$ . Let  $I_{\alpha+1} = q_{I_\alpha}^{-1}(P(A/I_\alpha))$ . Then  $I_{\alpha+1}/I_\alpha$  is  $P$ -radical and

$$I_{\alpha+1}/I_\alpha = P(A/I_\alpha) \subset (\vee \mathcal{F}_u)(A/I_\alpha) = (\vee \mathcal{F}_u)(A)/I_\alpha = I/I_\alpha$$

by Proposition 4.17(1b). So  $I_{\alpha+1} \subset I$ .

(2) Let  $J = (\wedge \mathcal{F}_o)(A)$ . Assume, by induction, that we already have the required chain  $(J_{\alpha'})_{\alpha' \leq \alpha}$  but with  $J \subset J_\alpha$ . If  $J_\alpha \neq J$  then  $J_\alpha/J$  is a non-zero  $\wedge \mathcal{F}_o$ -semisimple algebra by Proposition 4.17(1a). Then  $H_{\mathcal{F}_o}(J_\alpha/J) = 0$ , and there is some  $P \in \mathcal{F}$  such that  $P((J_\alpha/J)) \neq J_\alpha/J$ . Let  $J_{\alpha+1} = q_J^{-1}(P((J_\alpha/J)))$ . Then  $J_\alpha/J_{\alpha+1} \cong (J_\alpha/J)/P(J_\alpha/J)$  is  $P$ -semisimple.  $\square$

Let  $\mathfrak{U}$  be a base class, and let  $V$  be a class of [normed] algebras from  $\mathfrak{U}$ . We say that  $V$  is *closed under increasing transfinite chains* if  $V$  contains any [normed] algebra  $A$  for which there is an increasing transfinite chain  $(I_\alpha)_{\alpha \leq \gamma}$  of [closed] ideals  $A$  such that  $I_0 = 0$ ,  $I_\gamma = A$  and all gap-quotients of the chain are algebras from  $V$ ; we say that  $V$  is *closed under decreasing transfinite chains* if  $V$  contains any [normed] algebra  $A$  for which there is a decreasing transfinite chain  $(J_\alpha)_{\alpha \leq \delta}$  of [closed] ideals of  $A$  such that  $J_0 = A$ ,  $J_\delta = 0$  and all gap-quotients of the chain are algebras from  $V$ .

**Corollary 4.20.** *Let  $V$  be a class of [normed] algebras from  $\mathfrak{U}$ . Then*

- (1) *If  $V$  is closed under increasing transfinite chains then there is the largest radical on  $\mathfrak{U}$  whose radical algebras lie in  $V$ ;*
- (2) *If  $V$  is closed under decreasing transfinite chains then there is the smallest radical on  $\mathfrak{U}$  whose semisimple algebras lie in  $V$ .*

*Proof.* (1) Let  $\mathcal{F}$  be the family of all radicals  $P$  on  $\mathfrak{U}$  with  $\mathbf{Rad}(P) \subset V$ . Let  $R = \vee \mathcal{F}$ , and let  $A$  be an  $R$ -radical algebra. By Theorem 4.19, there is an increasing transfinite chain  $(I_\alpha)_{\alpha \leq \gamma}$  of [closed] ideals of  $A$  such that  $I_0 = 0$ ,  $I_\gamma = A$  and every gap-quotient of the chain is  $P$ -radical for some  $P \in \mathcal{F}$ . Then all gap-quotients of the chain lie in  $V$ , and  $R \in \mathcal{F}$ .

(2) The proof is similar to the proof of (1).  $\square$

A transfinite sequence  $(S_\alpha)_{\alpha \leq \delta}$  of over radicals is called a *decreasing transfinite chain* of over radicals if  $(S_\alpha(A))_{\alpha \leq \delta}$  is a decreasing transfinite chain of ideals of  $A$  for every algebra  $A$ ; similarly, a transfinite sequence  $(T_\alpha)_{\alpha \leq \gamma}$  of under radicals is called an *increasing transfinite chain* of under radicals if  $(T_\alpha(A))_{\alpha \leq \gamma}$  is an increasing transfinite chain of ideals of  $A$  for every algebra  $A$ .

Examples are:

- (1) A *convolution chain*  $(T_\alpha)_{\alpha \leq \gamma}$  obtained from family  $\mathcal{F}_u$  of under radicals by the rule  $T_0 := P_0 : A \mapsto 0$ , the zero radical, and  $T_{\alpha+1} = P * T_\alpha$  for some  $P \in \mathcal{F}_u$ ;
- (2) A *superposition chain*  $(S_\alpha)_{\alpha \leq \delta}$  obtained from a family  $\mathcal{F}_o$  of over radicals by the rule  $S_0 := P_1 : A \mapsto A$ , the identity radical, and  $S_{\alpha+1} = R \circ S_\alpha$  for some  $R \in \mathcal{F}_o$ .

Theorem 4.1 and Corollary 4.11 guarantee that  $(T_\alpha)_{\alpha \leq \gamma}$  consists of over radicals and  $(S_\alpha)_{\alpha \leq \delta}$  consists of under radicals.

**Theorem 4.21.** *Let  $\mathcal{F}_u^*$  be the family of all convolution chains of a family  $\mathcal{F}_u$  of under radicals, and let  $\mathcal{F}_o^\circ$  be the family of all superposition chains of a family  $\mathcal{F}_o$  of over radicals. Then*

- (1)  $T_\gamma \leq \vee \mathcal{F}_u$  for every  $(T_\alpha)_{\alpha \leq \gamma} \in \mathcal{F}_u^*$  and  $S_\delta \geq \wedge \mathcal{F}_o$  for every  $(S_\alpha)_{\alpha \leq \delta} \in \mathcal{F}_o^\circ$ .
- (2) For any algebras  $A_1, \dots, A_n$ , there exist
  - (a)  $(T_\alpha)_{\alpha \leq \gamma} \in \mathcal{F}_u^*$  such that  $T_\gamma(A_i) = (\vee \mathcal{F}_u)(A_i)$ , for  $i \leq n$ ;
  - (b)  $(S_\alpha)_{\alpha \leq \delta} \in \mathcal{F}_o^\circ$  such that  $S_\delta(A_i) = (\wedge \mathcal{F}_o)(A_i)$ , for  $i \leq n$ .

*Proof.* (1) follows by transfinite induction: the step  $\alpha \mapsto \alpha + 1$  follows by Lemma 4.14 and the case of limit ordinals is obvious.

(2a) Construct  $(T_\alpha) \in \mathcal{F}_u^*$  taking  $T_{\alpha+1} = P * T_\alpha$  for some  $P \in \mathcal{F}_u$  with  $P(A_1/T_\alpha(A_1)) \neq 0$ . The chain  $(T_\alpha(A_1))$  of ideals of  $A_1$  must be stabilized: there is  $T_{\gamma_1}$  such that  $P(A_1/T_{\gamma_1}(A_1)) = 0$  for every  $P \in \mathcal{F}_u$ . Continuing the construction of  $(T_\alpha)_{\alpha \leq \gamma_1}$  first for  $A_2, \dots, A_n$  and then for every other algebra  $B$  in such a way if necessary, one can find  $(T_\alpha)_{\alpha \leq \gamma_i}$  with  $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_n = \gamma$  such that  $P(A_i/T_{\gamma_i}(A_i)) = 0$  for every  $P \in \mathcal{F}_u$ , and define a preradical  $T$  such that  $T(A_i) = T_{\gamma_i}(A_i)$  for  $i = 1, \dots, n$ . By Theorem 4.1 and Corollary 4.11,  $T$  is an under radical and  $T \leq \vee \mathcal{F}_u$  by (1). It is clear that  $T(B/T(B)) = 0$ , i.e.  $T$  is a radical. As

$$q_{T(B)}(P(B)) \subset P(B/T(B)) = 0$$

for every  $P \in \mathcal{F}_u$ , then  $P(B) \subset T(B)$ , for every algebra  $B$ . Hence  $\vee \mathcal{F}_u \leq T$ .

(2b) Construct  $(S_\alpha) \in \mathcal{F}_o^\circ$  taking  $S_{\alpha+1}(A_1) = R(S_\alpha(A_1)) \neq S_\alpha(A_1)$  for some  $R \in \mathcal{F}_o$ . The chain  $(S_\alpha(A_1))$  of ideals must be stabilized: there is  $S_{\delta_1}$  such that  $R(S_{\delta_1}(A_1)) = S_{\delta_1}(A_1)$  for every  $R \in \mathcal{F}_o$ . Continuing the construction of  $(S_\alpha)$  first for  $A_2, \dots, A_n$  and then for every other algebra  $B$  in such a way if necessary, one

can find  $(S_\alpha)_{\alpha \leq \delta_i}$  with  $\delta_1 \leq \delta_2 \leq \dots \leq \delta_n = \delta$  such that  $R(A_i/S_{\delta_i}(A_i)) = 0$  for every  $R \in \mathcal{F}_o$ , and define a preradical  $S$  such that  $S(A_i) = S_\delta(A_i)$  for  $i = 1, \dots, n$ . By Theorem 4.1 and Corollary 4.11,  $S$  is an over radical and  $S \geq \wedge \mathcal{F}_o$  by (1). It is clear that  $S(S(B)) = S(B)$ , i.e.  $S$  is a radical. As  $S(B) = R(S(B)) \subset R(B)$  for every  $R \in \mathcal{F}_o$  and every algebra  $B$ , then  $S \leq \wedge \mathcal{F}_o$ .  $\square$

**4.4. The heredity problem.** The closure procedure does not preserve the class of hereditary radicals:  $\overline{P}$  can be non-hereditary when  $P$  is a hereditary preradical (see [D3, Example 7.1]). The operation of the closure (of an ideal) is involved also in the construction of  $P^*$ ; this leads to a possible loss of heredity. The following lemma shows that it is not the case for the convolution chains in  $\mathfrak{U}_a$  (in particular, for the algebraic convolution procedure) and for the superposition chains in general.

**Lemma 4.22.** *Let  $\delta$  be a limit ordinal. Then*

- (1) *If  $(P_\alpha)_{\alpha \leq \delta}$  is a decreasing transfinite chain of preradicals and  $P_\alpha$  is hereditary for  $\alpha < \delta$ , then  $P_\delta$  is hereditary.*
- (2) *If  $(P_\alpha)_{\alpha \leq \delta}$  is an increasing transfinite chain of algebraic preradicals and  $P_\alpha$  is hereditary for  $\alpha < \delta$ , then  $P_\delta$  is hereditary.*

*Proof.* (1) Let  $A$  be an algebra, and let  $J$  be an ideal of  $A$ . By definition, the chain  $(P_\alpha(A))_{\alpha \leq \delta}$  is a decreasing transfinite chain of [closed] ideals. Then

$$P_\delta(A) = \cap_{\alpha < \delta} P_\alpha(A)$$

by (3.1), and  $P_\delta(J) = J \cap P_\delta(A)$ .

- (2) By definition,  $(P_\alpha(A))_{\alpha \leq \delta}$  is an increasing transfinite chain of ideals. So

$$\begin{aligned} P_\delta(J) &= \cup_{\alpha < \delta} P_\alpha(J) = \cup_{\alpha < \delta} (J \cap P_\alpha(A)) = J \cap (\cup_{\alpha < \delta} P_\alpha(A)) \\ &= J \cap P_\delta(A). \end{aligned}$$

$\square$

**Corollary 4.23.** *If  $P$  is an algebraic hereditary preradical then  $P^*$  is a hereditary radical.*

*Proof.* It follows from Theorem 4.11 and Lemma 4.22 that the convolution chain generated by  $P$  consists of hereditary preradicals. Therefore  $P^*$  is hereditary.  $\square$

Let  $\mathcal{F}$  be a family of hereditary radicals. Then  $\wedge \mathcal{F}$  is a hereditary radical and therefore is the infimum of  $\mathcal{F}$  in the class of hereditary radicals. Besides of this  $\wedge \mathcal{F}$  satisfies the following equality [ST7, Lemma 3.2]

$$\wedge \mathcal{F} = B_{\mathcal{F}}. \quad (4.2)$$

**Theorem 4.24.** *The operations  $\vee$  and  $\wedge$  restricted to the class of algebraic hereditary radicals produce supremum and infimum in this class; moreover  $\wedge$  produces infimum in the class of topological hereditary radicals.*

*Proof.* Let  $\mathcal{F}$  be a family of algebraic hereditary radicals,  $A$  an algebra, and let  $J$  be an ideal of  $A$ . By Theorem 4.21, there is a convolution chain  $(T_\alpha)_{\alpha \leq \gamma}$  obtained from  $\mathcal{F}$  such that  $(\vee \mathcal{F})(A) = T_\gamma(A)$  and  $(\vee \mathcal{F})(J) = T_\gamma(J)$ . By Theorem 4.11 and Lemma 4.22,  $(T_\alpha)_{\alpha \leq \gamma}$  consists of hereditary preradicals. Therefore  $T_\gamma(J) = J \cap T_\gamma(A)$ . This shows that  $\vee \mathcal{F}$  is a hereditary radical. Therefore  $\vee \mathcal{F}$  is the supremum of  $\mathcal{F}$  in the class of algebraic hereditary radicals.

The assertion about the infimum is evident.  $\square$

**Theorem 4.25.** *Let  $\mathcal{F}$  be a family of hereditary radicals and  $V = \cup_{P \in \mathcal{F}} \mathbf{Rad}(P)$ . Let  $\mathcal{F}'$  be the family of all hereditary radicals  $R$  such that  $\mathbf{Rad}(R) \supset V$ . Then the supremum of  $\mathcal{F}$  in the class of hereditary radicals is equal to*

$$\wedge \mathcal{F}' = B_{\mathcal{F}'}.$$

*Proof.* Let  $T = \wedge \mathcal{F}'$ . Then  $T$  is a hereditary radical by [ST7, Lemma 3.2], and  $T \leq R$ , for every  $R \in \mathcal{F}'$ , by definition. As  $\wedge \mathcal{F}' = B_{\mathcal{F}'}$  then, for every  $P \in \mathcal{F}$ , we obtain that

$$\mathbf{Rad}(T) = \cap_{R \in \mathcal{F}'} \mathbf{Rad}(R) \supset V \supset \mathbf{Rad}(P)$$

and therefore  $T \geq P$ . So  $T$  is the supremum of  $\mathcal{F}$  in the class of hereditary radicals.  $\square$

Now we consider a way to show heredity of some radicals that are constructed via the closure procedure and the topological convolution procedure.

Let  $A$  be an algebra, and let  $I$  be an ideal of  $A$ . Let  $I^\perp = \{x \in A : xI = 0\}$ , the left annihilator of  $I$  in  $A$ . A preradical  $P$  is called *nil-exact* if

$$I^\perp (P * I) \subset P(A)$$

for every [closed] ideal  $I$  of any algebra  $A$ .

A preradical  $P$  is called *weakly hereditary* if, for any  $P$ -radical algebra  $A$ ,  $P(I) \neq 0$  for every non-zero ideal  $I$  of  $A$ .

Recall that  $\Sigma_\beta(A)$  means the sum of all nilpotent ideals of  $A$ ; if  $P$  is a topological preradical then  $\Sigma_\beta \leq P$  means that the inequality holds for normed algebras on which  $P$  is defined.

**Lemma 4.26.** *Let  $P$  be an algebraic under radical. Then*

- (1) *If  $P$  is nil-exact then  $\overline{P}$  is nil-exact;*
- (2) *If  $P$  is weakly hereditary and  $\Sigma_\beta \leq P$  then  $\overline{P}$  is weakly hereditary.*

*Proof.* (1) Let  $A$  be a normed algebra, and let  $I$  be a closed ideal of  $A$ . We have that  $I^\perp (P * I) \subset P(A)$ , i.e.  $I^\perp q_I^{-1}(P(A/I)) \subset P(A)$ . By (1.5),

$$q_I^{-1}(\overline{P(A/I)}) = q_I^{-1}(\overline{P(A/I)}) = \overline{q_I^{-1}(P(A/I))},$$

so

$$I^\perp q_I^{-1}(\overline{P(A/I)}) = I^\perp \overline{q_I^{-1}(P(A/I))} \subset \overline{I^\perp q_I^{-1}(P(A/I))} \subset \overline{P(A)} = \overline{P}(A),$$

i.e.  $I^\perp (\overline{P} * I) \subset \overline{P}(A)$ .

(2) Let  $A$  be a  $\overline{P}$ -radical algebra, and let  $I$  be a non-zero ideal of  $A$ . Set  $J = IA$ .

If  $J \neq 0$  then  $P(J) \neq 0$  by the weak heredity of  $P$ . As  $J$  is an ideal of  $I$ , it follows that  $P(I) \neq 0$ , whence  $\overline{P}(I) \neq 0$ .

If  $IA = 0$  then  $I^2 = 0$ , whence  $I = \Sigma_\beta(I) \subset P(I)$  and  $\overline{P}(I) = I \neq 0$ .  $\square$

**Lemma 4.27.** *Let  $P$  be a weakly hereditary under radical, and let  $A$  be an algebra and  $I$  an ideal of  $A$ . Then*

- (1) *If  $I \cap P(A) \neq 0$  then  $P(I) \neq 0$ ;*
- (2) *If  $I \subset P(A)$  then  $I$  is  $P^*$ -radical.*

*Proof.* (1) Let  $J = I \cap P(A) \neq 0$ . Then  $J$  is a non-zero ideal of the  $P$ -radical algebra  $P(A)$ , whence  $P(J) \neq 0$  by weak heredity. As  $J$  is an ideal of  $I$  then  $P(J) \subset P(I) \neq 0$ .

(2) Let  $A$  be an algebra, and let  $I$  be a non-zero ideal of  $A$  and  $I \subset P(A)$ . Then  $I$  is an ideal of the  $P$ -radical algebra  $B_1 := P(A)$ . Let  $I_1 = P * 0 = P(I)$  with respect to the algebra  $I$ ; then  $I_1 \neq 0$  is an ideal of  $B_1$ . Let  $I'_1$  be  $I_1$  for the algebraic case and the closure of  $I_1$  in  $B_1$  for the topological case. Then  $B_1/I'_1 = q_{I'_1}(P(A)) \subset B_2 := P(A/I'_1)$  and  $I/I_1 \cong q_{I'_1}(I)$ .

If  $I/I_1 \neq 0$  then  $q_{I'_1}(I)$  is a non-zero ideal of the  $P$ -radical algebra  $B_2$ . Let  $I_2 = P * I_1$  with respect to the algebra  $I$ :  $I_2 = q^{-1}(P(I/I_1))$  where  $q : I \rightarrow I/I_1$  is the standard quotient map. If  $q_{I'_1}(I) \neq 0$  then  $P(q_{I'_1}(I)) \neq 0$  by the assumptions and therefore  $I_2 \neq I_1$ .

Using transfinite induction, we can build the increasing transfinite chain of ideals  $(I_\alpha)$  of ideals of  $A$  such that all  $I_\alpha \subset I$  and  $I_{\alpha+1} = P * I_\alpha \neq I_\alpha$  if  $I \neq I_\alpha$ , and also the corresponding transfinite chain  $(I'_\alpha)$  (consisting of closures of  $I_\alpha$ 's for the topological case) such that  $q_{I'_\alpha}(I)$  is an ideal of the  $P$ -radical algebra  $P(A/I'_\alpha)$ . The chain  $(I_\alpha)$  must be stabilized: there is an ordinal  $\gamma$  such that  $I = I_\gamma$ . By construction,  $I$  is  $P^*$ -radical.  $\square$

**Lemma 4.28.** *Let  $P$  be a nil-exact and weakly hereditary under radical, and let  $I$  be an ideal of an algebra  $A$ . If  $IP^*(A) \neq 0$  then  $P(I) \neq 0$ .*

*Proof.* If  $IP(A) \neq 0$  then  $P(I) \neq 0$  by Lemma 4.27(1). So one may assume that  $IP(A) = 0$ . For the convolution chain  $(P_\alpha)$  (with  $P_0 = \mathcal{P}_0$  and  $P_{\alpha+1} = P * P_\alpha$  for all  $\alpha$ ), one can find the first ordinal  $\gamma$  such that  $I(P * P_\gamma(A)) \neq 0$  (this is possible because it is clear that if  $IP_{\alpha'}(A) = 0$  for  $\alpha' < \alpha$  and  $\alpha$  is a limit ordinal then  $IP_\alpha(A) = 0$ ). Let  $J = P_\gamma(A)$ . Then  $Iq_J^{-1}(P(A/J)) \neq 0$  and  $IJ = 0$  by the choice of  $\gamma$ . By nil-exactness of  $P$ ,  $Iq_J^{-1}(P(A/J)) \subset P(A)$ . As  $Iq_J^{-1}(P(A/J)) \subset I$  then  $I \cap P(A) \neq 0$ , whence  $P(I) \neq 0$  by Lemma 4.27(1).  $\square$

**Lemma 4.29.** *Let  $P$  be a weakly hereditary and nil-exact under radical, and  $\Sigma_\beta \leq P$ . Then  $P^*$  is weakly hereditary.*

*Proof.* Let  $A$  be a  $P^*$ -radical algebra, and let  $I$  be a non-zero ideal of  $A$ . If  $I^2 = 0$  then  $I = \Sigma_\beta(I) \subset P(I) \subset P^*(I)$  and  $P^*(I) = I \neq 0$ . If  $I^2 \neq 0$  then  $IP^*(A) \neq 0$  and  $P(I) \neq 0$  by Lemma 4.28. Therefore  $P(I) \subset P^*(I) \neq 0$ .  $\square$

**Theorem 4.30.** *Let  $P$  be a weakly hereditary and nil-exact under radical, and  $\Sigma_\beta \leq P$ . Then  $P^*$  is hereditary.*

*Proof.* Let  $A$  be an algebra, and let  $I$  be an ideal of  $A$ . By Lemma 4.29,  $P^*$  is weakly hereditary. We apply Lemma 4.27(2) to  $P^*$  instead of  $P$ . As  $P^*$  is a radical (in particular,  $P^{**} = P^*$ ), every ideal  $J$  of  $A$  with  $J \subset P^*(A)$  is  $P^*$ -radical. Then  $J := I \cap P^*(A)$  is  $P^*$ -radical; as  $J$  is an ideal of  $I$ , it follows that

$$J = P^*(J) \subset P^*(I) \subset I \cap P^*(A) = J,$$

i.e.  $P^*(I) = I \cap P^*(A)$ . Therefore  $P^*$  is hereditary.  $\square$

As is well known (see for instance [AR]), the radicals  $\mathfrak{P}_\beta$ ,  $\mathfrak{P}_\lambda$ ,  $\mathfrak{P}_\kappa$  and  $\text{rad}$  are hereditary.

**Theorem 4.31.**  *$\mathcal{P}_\beta$ ,  $\mathcal{P}_\lambda$ ,  $\mathcal{P}_\kappa$  and  $\overline{\text{rad}}^*$  are hereditary.*

*Proof.* To apply Lemma 4.26 and Theorem 4.30, one has to show that the algebraic under radicals  $\Sigma_\beta$  (see Corollary 3.7),  $\mathfrak{P}_\lambda$ ,  $\mathfrak{P}_\kappa$  and  $\text{rad}$  are weakly hereditary and nil exact. Since any ideal of a  $\Sigma_\beta$ -radical, locally nilpotent, nil or Jacobson-radical algebra have the same structure, all the under radicals are weakly hereditary. It remains to establish nil exactness. Let  $A$  be an algebra, and let  $I$  be an ideal of  $A$ ,  $B = A/I$  and  $a \in I^\perp$ .

Let  $x \in q_I^{-1}(\Sigma_\beta(B))$ ; then  $q_I(x) = b_1 + \dots + b_n$ , where  $b_i$  is in a nilpotent ideal of  $B$ , for each  $i$ . We take  $x_i \in A$  such that  $b_i = q_I(x_i)$ . Then  $q(ax_i)$  lies in a nilpotent ideal of  $B$  for every  $i$ : there is  $n_i > 0$  such that  $q_I(ax_i A)^{n_i} = 0$ , i.e.  $(ax_i A)^{n_i} \subset I$ . As  $ax_i A \subset I^\perp = 0$  then  $(ax_i A)^{n_i+1} = 0$ . Hence  $ax_i \in \Sigma_\beta(A)$  for every  $i$  and therefore  $ax = ax_1 + \dots + ax_n \in \Sigma_\beta(A)$ .

Let  $N \subset q_I^{-1}(\mathfrak{P}_\lambda(B))$  be finite, and let  $A_{aN}$  be the subalgebra generated by  $aN$ . Then  $q_I(aN) \subset \mathfrak{P}_\lambda(B)$  generates the nilpotent subalgebra: there is  $m > 0$  such that  $q_I(A_{aN})^m = 0$ , i.e.  $(A_{aN})^m \subset I$ . But  $A_{aN} \subset I^\perp$ , whence  $(A_{aN})^{m+1} = 0$ . As  $N$  is arbitrary, we conclude that  $aq_I^{-1}(\mathfrak{P}_\lambda(B)) \subset \mathfrak{P}_\lambda(A)$ .

Let  $x \in q_I^{-1}(\mathfrak{P}_\kappa(B))$ , and let  $J$  be the ideal of  $A$  generated by  $ax$ . Then  $q_I(J)$  consists of nilpotents. In particular, if  $b \in J$  is arbitrary then there is  $k > 0$  such that  $b^k \in I$ . As  $b \in I^\perp$  then  $b^{k+1} = 0$ . This means that  $ax \in \mathfrak{P}_\kappa(A)$ .

Let  $x \in q_I^{-1}(\text{rad}(B))$ , and let  $\pi \in \text{Irr}(A)$  be arbitrary. If  $\pi(I) \neq 0$  then  $\pi(I)$  is a strictly irreducible algebra of operators and  $\pi(a)\pi(I) = \pi(aI) = 0$  implies that  $\pi(a) = 0$ , whence  $\pi(ax) = 0$ . If  $\pi(I) = 0$  then there exists  $\tau \in \text{Irr}(B)$  such that  $\pi = \tau \circ q_I$ ; as  $\tau(q_I(ax)) = 0$  then  $\pi(ax) = 0$ . Therefore  $ax \in \text{rad}(A)$ .

All under radicals in consideration are nil exact. By Lemma 4.26 and Theorem 4.30,  $\mathcal{P}_\beta$ ,  $\mathcal{P}_\lambda$ ,  $\mathcal{P}_\kappa$  and  $\overline{\text{rad}}^*$  are hereditary.  $\square$

Establishing heredity of the radical  $\mathcal{P}_\beta$ , Theorem 4.31 answers a question posed by Dixon [D3, Page 188].

Recall that, for any algebra  $A$ ,  $F(A)$  is the set of all finite rank elements of  $A$ .

**Lemma 4.32.** *Let  $A$  be a semiprime algebra. Then*

- (1)  $F(A) \cap \text{rad}(A) = 0$ .
- (2) *If  $A$  is normed then  $\overline{F(A)} \cap \overline{\text{rad}(A)} = 0$ .*

*Proof.* (1) Assume that  $x \in F(A) \cap \text{rad}(A)$  is nonzero, and let  $n$  be the rank of  $W_x$ . Then  $xAx$  is  $n$ -dimensional and the subalgebra generated by  $x$  in  $A^1$  is at most  $(n+2)$ -dimensional. So  $x$  is an algebraic element in  $\text{rad}(A)$ , whence there is a minimal polynomial  $p(x)$  on  $x$  with degree at most  $n+2$ . The roots of  $p$  lie in the spectrum of  $x$  with respect to  $A^1$  by [Bo, Section 1.1.2]. As  $x$  is in  $\text{rad}(A)$ , these roots are equal to 0, whence  $x^{n+2} = 0$ .

Let  $a \in A^1$  be arbitrary. As  $W_{ax} = L_a W_x R_a$ , the rank of  $W_{ax}$  is at most  $n$ . As  $ax \in \text{rad}(A)$ , it follows that  $(ax)^{n+2} = 0$  by above. Since  $A^1 x$  consists of nilpotents of fixed degree,  $A^1 x$  is a nilpotent algebra by the Nagata-Higman theorem (in fact, Dubnov-Nagata-Higman theorem, see [Fr]). As  $A^1 x$  is a non-zero left nilpotent ideal of  $A$ , we obtain that  $A$  is not semiprime, a contradiction. Therefore  $F(A) \cap \text{rad}(A) = 0$ .

(2) Let  $x, y \in I := \overline{F(A)} \cap \overline{\text{rad}(A)}$  be arbitrary. Let  $(a_n) \subset F(A)$  and  $(b_n) \subset \text{rad}(A)$  be such that  $a_n \rightarrow x$  and  $b_n \rightarrow y$  as  $n \rightarrow \infty$ . Then  $a_n b_n = 0$  for all  $n$  by (1), but  $a_n b_n \rightarrow xy$  as  $n \rightarrow \infty$ , whence  $xy = 0$  and  $I^2 = 0$ . As  $I$  is an ideal of  $A$  and  $A$  is semiprime,  $I = 0$ .  $\square$

Let us, for brevity, call a subset  $M$  of an algebra  $A$  *square zero* if  $ab = 0$  for all  $a, b \in M$ .

**Theorem 4.33.**  $\mathfrak{P}_\beta = \mathfrak{R}_{\text{hf}} \wedge \text{rad}$  and  $\mathcal{R}_\beta = \mathcal{R}_{\text{hf}} \wedge \overline{\text{rad}}^* \leq \mathcal{R}_{\text{jhc}} = \mathcal{R}_{\text{hc}} \wedge \mathcal{R}_{\text{cq}}$ .

*Proof.* Let  $A$  be an algebra, and let  $\Sigma_0(A)$  denote the sum of all square zero left ideals of  $A$ . If  $L$  is a square zero left ideal of  $A$  then  $La$  is also a square zero left ideal of  $A$ :  $L(aL)a \subset (LL)a = 0$ . So  $\Sigma_0(A)$  is an ideal of  $A$ . It is easy to check that  $\Sigma_0$  is an under radical. If  $I$  is a nilpotent ideal of  $A$  then it is obvious that  $I \subset \Sigma_0^*(A)$ . Therefore  $\Sigma_\beta(A) \subset \Sigma_0^*(A)$ . As  $\Sigma_0^*$  is a radical then  $\mathfrak{P}_\beta = \Sigma_\beta^* \subset \Sigma_0^*$ . On the other hand, every semiprime algebra is  $\Sigma_0^*$ -semisimple, whence

$$\mathfrak{P}_\beta = \Sigma_0^*.$$

Note that every square zero left ideal consists of zero rank elements of  $A$ . So  $\Sigma_0(A)$  is an ideal of  $A$  consisting of finite rank elements. Therefore  $\Sigma_0 \leq \mathfrak{R}_{\text{hf}}$ , whence

$$\mathfrak{P}_\beta = \Sigma_0^* \leq \mathfrak{R}_{\text{hf}}.$$

It is well-known that  $\mathfrak{P}_\beta \leq \text{rad}$ . Thus

$$\mathfrak{P}_\beta \leq \mathfrak{R}_{\text{hf}} \wedge \text{rad}.$$



Let now  $A$  be semiprime. We must to prove that  $A$  is  $(\mathfrak{R}_{\text{hf}} \wedge \text{rad})$ -semisimple. Assume, to the contrary, that

$$I := (\mathfrak{R}_{\text{hf}} \wedge \text{rad})(A) = \mathfrak{R}_{\text{hf}}(A) \cap \text{rad}(A) \neq 0$$

by (4.2). By heredity of  $\mathfrak{R}_{\text{hf}}$ ,  $I$  is an  $\mathfrak{R}_{\text{hf}}$ -radical ideal of  $A$ . By Theorem 2.20(5), there is a non-zero finite rank element  $x$  of  $A$  such that  $x \in \text{rad}(A)$ , a contradiction to Lemma 4.32. So  $A$  is  $(\mathfrak{R}_{\text{hf}} \wedge \text{rad})$ -semisimple, whence

$$\mathfrak{R}_{\text{hf}} \wedge \text{rad} \leq \mathfrak{P}_\beta.$$

It is clear that  $\overline{\mathfrak{P}_\beta}^* \leq \overline{\mathfrak{R}_{\text{hf}}}^* \wedge \overline{\text{rad}}^*$ , whence  $\mathcal{P}_\beta \leq \mathcal{R}_{\text{hf}} \wedge \overline{\text{rad}}^*$ .

Let  $A$  be a  $\mathcal{P}_\beta$ -semisimple normed algebra; then  $A$  is semiprime. Assume, to the contrary, that  $I := (\mathcal{R}_{\text{hf}} \wedge \overline{\text{rad}}^*)(A) \neq 0$ . By Corollary 4.16,  $I = \overline{\text{rad}}^*(\mathcal{R}_{\text{hf}}(A))$ . Therefore  $J := \overline{\text{rad}}(\mathcal{R}_{\text{hf}}(A)) \neq 0$ . Arguing as above, one has that there is a non-zero element  $x \in \overline{F}(A)$  such that  $x \in J \subset \overline{\text{rad}}(A)$ , whence  $x = 0$  by Lemma 4.32, a contradiction. Hence  $A$  is  $(\mathcal{R}_{\text{hf}} \wedge \overline{\text{rad}}^*)$ -semisimple. Therefore  $\mathcal{P}_\beta \leq \mathcal{R}_{\text{hf}} \wedge \overline{\text{rad}}^* \leq \mathcal{P}_\beta$ .

By definition,  $\mathcal{R}_{\text{jhc}} = \mathcal{R}_{\text{hc}} \wedge \text{Rad}^r$ . By (2.9),  $\mathcal{R}_{\text{hc}} \wedge \text{Rad}^r \leq \mathcal{R}_{\text{cq}}$ , whence

$$\mathcal{R}_{\text{jhc}} = (\mathcal{R}_{\text{hc}} \wedge \text{Rad}^r) \wedge \mathcal{R}_{\text{cq}} = \mathcal{R}_{\text{hc}} \wedge \mathcal{R}_{\text{cq}}.$$

As every nilpotent ideal of  $A$  lies in  $\mathcal{R}_{\text{cq}}(A)$  then  $\mathfrak{P}_\beta \leq \mathcal{R}_{\text{cq}}$ , whence also  $\mathcal{P}_\beta \leq \mathcal{R}_{\text{cq}}$ . Taking into consideration that  $\mathcal{P}_\beta = \mathcal{R}_{\text{hf}} \wedge \overline{\text{rad}}^* < \mathcal{R}_{\text{hc}}$ , we obtain that  $\mathcal{P}_\beta \leq \mathcal{R}_{\text{jhc}}$ .  $\square$

## 5. CENTRALIZATION

**5.1. Commutative ideals and centralization of radicals.** Let  $A \in \mathfrak{U}_a$  be an algebra, and let  $Z(A)$  be the center of  $A$ . An ideal  $J$  of  $A$  is called *central* if  $J \subset Z(A)$ . Let  $\Sigma_a(A)$  be the sum of all commutative ideals of  $A$ .

**Lemma 5.1.** (1)  $\Sigma_a$  is a preradical.

(2) If  $A$  is a semiprime algebra then  $\Sigma_a(A)$  is the largest central ideal of  $A$  and

- (a)  $\Sigma_a(A) = \{x \in A : x[a, b] = 0 \ \forall a, b \in A\}$ ;
- (b)  $\Sigma_a(J) = J \cap \Sigma_a(A)$  for every ideal  $J$  of  $A$ .

*Proof.* (1) is straightforward.

(2) Let  $J$  be a commutative ideal of  $A$ , and let  $x, y \in J$ ,  $a, b \in A$  be arbitrary. Then

$$[x, a]y = x(ay) - axy = (ay)x - axy = a[y, x] = 0.$$

As  $b[x, a] \in J$ , it follows that  $[x, a]b[x, a] = 0$ , i.e.  $[x, a]A[x, a] = 0$ . As  $A$  is semiprime,  $[x, a] = 0$ , i.e.  $J \subset Z(A)$ , whence  $\Sigma_a(A) \subset Z(A)$ . So  $\Sigma_a(A)$  is a commutative ideal of  $A$ . Then, by definition,  $\Sigma_a(A)$  is the largest ideal that lies in  $Z(A)$ .

(a) Let  $K = \{x \in A : x[a, b] = 0 \ \forall a, b \in A\}$ . Let  $x \in K$ ,  $a, b, c \in A$  be arbitrary. It is clear that  $K$  is an ideal of  $A$ : indeed, one has

$$xa[b, c] = xa[b, c] + x[a, c]b = x[ab, c] = 0.$$

Then  $[x, a]b \in K$ , whence  $[x, a]b[x, a] = 0$ . Since  $A$  is semiprime,  $[x, a] = 0$ , i.e.  $x \in Z(A)$ . Therefore  $K \subset \Sigma_a(A)$ .

Let  $y \in \Sigma_a(A)$  be arbitrary. As  $y, ya \in Z(A)$ , we obtain that

$$y[a, b] = [ya, b] - [y, b]a = 0.$$

Thus  $\Sigma_a(A) \subset K$ .

(b) Obviously  $J \cap \Sigma_a(A) \subset \Sigma_a(J)$ . As  $J$  is a semiprime algebra,  $\Sigma_a(J)$  is described as in (a) for  $A = J$ . Then  $\Sigma_a(J)$  is a left ideal of  $A$  that lies in the center

of  $J$ . By symmetry,  $\Sigma_a(J)$  is a commutative ideal of  $A$  and so  $\Sigma_a(J) \subset \Sigma_a(A)$  by definition.  $\square$

**Remark 5.2.** *It follows from Lemma 5.1(2) that if  $A$  is a semiprime normed algebra then  $\Sigma_a(A)$  is closed.*

Let  $P$  be a preradical and let

$$P^a = \Sigma_a * P;$$

$P^a$  is called the *centralization* of  $P$ . The map  $P \mapsto P^a$  is called the *centralization procedure*.

Let  $P$  be a radical such that  $P \geq \mathfrak{P}_\beta$ . If  $P$  is topological on  $\mathfrak{U}_n$  or  $\mathfrak{U}_b$ , the inequality  $P \geq \mathfrak{P}_\beta$  is assumed to keep on  $\mathfrak{U}_n$  or  $\mathfrak{U}_b$ , respectively; in such a case the inequality  $P \geq \mathcal{P}_\beta$  also holds.

**Theorem 5.3.** *Let  $P$  be a radical such that  $P \geq \mathfrak{P}_\beta$ . Then*

- (1)  $P^a$  is an under radical;
- (2) If  $P$  is hereditary then  $P^a$  is hereditary.

*Proof.* Note that  $P^a$  is a preradical by Lemma 4.10. Remark 5.2 guarantees that  $P^a$  is a topological preradical in the topological case.

Let  $A$  be an algebra, and let  $J$  be an ideal of  $A$ . We claim that  $A/P(A)$  is semiprime: indeed,  $\mathfrak{P}_\beta(A/P(A)) \subset P(A/P(A)) = 0$ . Similarly,  $J/P(J)$  is a semiprime algebra. It follows from Lemma 5.1(2a) that

$$\begin{aligned} P^a(A) &= \{x \in A : x[a, b] \in P(A) \ \forall a, b \in A\}, \\ P^a(J) &= \{x \in J : x[y, z] \in P(J) \ \forall y, z \in J\}. \end{aligned} \tag{5.1}$$

Let  $x \in P^a(J)$  be arbitrary, and let  $q : A/P(J) \rightarrow A/P(A)$  be the standard quotient map. Then

$$(q \circ q_{P(J)})(x) \in \Sigma_a(q(J/P(J))) \subset \Sigma_a(A/P(A))$$

by Lemma 5.1(2b), whence  $x \in P^a(A)$ . So

$$P^a(J) \subset P^a(A).$$

(2) Let  $x \in J \cap P(A)$ . Then  $x[a, b] \in J \cap P(A)$  for every  $a, b \in A$ , in particular for every  $a, b \in J$ . As  $J \cap P(A) = P(J)$ , it follows that  $x \in P^a(J)$ . Therefore,  $P^a(J) = J \cap P^a(A)$ . Thus  $P^a$  is a hereditary preradical, in particular, it is an under radical. This completes the proof of (2).

(1) It follows from (5.1) applied to the opposite algebra of  $A$  that  $P^a(J)$  is an ideal of  $A$ . It is clear that  $P(P^a(A)) \subset P(A) \subset P^a(A)$ , and then we obtain that  $P(A) = P(P(A)) \subset P(P^a(A))$ , whence

$$P(A) = P(P^a(A)). \tag{5.2}$$

It follows from Lemma 5.1(2) that  $[a, b] \in P(A)$  for every  $a, b \in P^a(A)$ , and from (5.1) for  $J = P^a(A)$  and (5.2) that

$$P^a(P^a(A)) = \{x \in P^a(A) : x[a, b] \in P(A) \ \forall a, b \in P^a(A)\} = P^a(A).$$

This completes the proof of (1).  $\square$

We see from Theorem 5.3 that in general  $P^a$  is an under radical (if  $P \geq \mathfrak{P}_\beta$ ), and one can apply the convolution procedure to obtain the corresponding radical  $P^{a*}$  (the *centralized  $P$  radical*). But  $P^{a*}$  has a much more complicated description than  $P^a$  which can be transparently defined by the condition:  $P^a(A)$  is the largest ideal of  $A$  commutative modulo  $P(A)$ .

Below we study the question: *when  $P^a$  itself is a radical?* Earlier the answer was obtained in the case of the compactly quasinilpotent radical  $R_{\text{cq}}$  — it was proved in [ST7] that  $\mathcal{R}_{\text{cq}}^a$  is a topological radical on  $\mathfrak{U}_n$ .

Theorem 5.3 and Lemma 4.7 immediately imply the following result.

**Lemma 5.4.** *Let  $P$  be a radical such that  $P \geq \mathfrak{P}_\beta$ . Then  $P^a$  is a radical if and only if, for every [closed] ideal  $J$  of an algebra  $A$  such that the algebras  $A/J$  and  $J$  are  $P^a$ -radical, the algebra  $A$  is  $P^a$ -radical.*

In the following proposition we obtain a tool for the proof that  $P^a$  is a radical, which turns out to be convenient for many important examples of radicals.

**Proposition 5.5.** *Let  $P$  be a radical such that  $P \geq \mathfrak{P}_\beta$ . Then  $P^a$  is a radical if and only if*

(\*) *any  $P$ -semisimple algebra  $B$  with  $P^a$ -radical  $B/\Sigma_a(B)$ , is commutative.*

*Proof.* Suppose that (\*) is true. By Lemma 5.4, it suffices to show that if a [closed] ideal  $J$  of an algebra  $A$  and the quotient  $A/J$  are  $P^a$ -radical then  $A$  is  $P^a$ -radical.

Let  $B = A/P(A)$  and  $I$  be the [closure of the] image of  $J$  under the standard map  $A \rightarrow A/P(A)$ . Then  $B$  is  $P$ -semisimple, in particular semiprime. Since  $J$  is commutative modulo  $P(J) \subset P(A)$  then  $I$  is a commutative ideal of  $B$ . Therefore  $I \subset \Sigma_a(B)$  and, using the standard quotient map  $q : A/J \rightarrow B/I \rightarrow B/\Sigma_a(B)$ , we get

$$B/\Sigma_a(B) = q(A/J) = q(P^a(A/J)) \subset P^a(B/\Sigma_a(B))$$

i.e.,  $B/\Sigma_a(B)$  is  $P^a$ -radical.

By (\*),  $B$  is commutative whence  $B = \Sigma_a(B)$  and

$$A = q_{P(A)}^{-1}(\Sigma_a(A/P(A))) = P^a(A),$$

i.e.  $A$  is  $P^a$ -radical. By Proposition 5.4,  $P^a$  is a radical.

Conversely, let  $P^a$  be a radical, and let  $A$  be a  $P$ -semisimple algebra such that  $A/\Sigma_a(A)$  is  $P^a$ -radical. Let now  $q : A/\Sigma_a(A) \rightarrow A/P^a(A)$  be the natural morphism. Since the first algebra is  $P^a$ -radical then its image is contained in  $P^a(A/P^a(A)) = 0$ . Therefore  $A/P^a(A) = 0$ , whence  $A = P^a(A)$ . Since  $P(A) = 0$ , we get that  $P^a(A) = \Sigma_a(A)$ . Thus  $A = \Sigma_a(A)$ , i.e.,  $A$  is commutative. We proved (\*).  $\square$

**Lemma 5.6.** *Let  $A$  be an algebra, and let  $J$  be an ideal of  $A$  such that  $\Sigma_a(A) \subset J$ . If  $A$  is semiprime and  $J/\Sigma_a(A)$  is commutative or nilpotent then  $J = \Sigma_a(A)$ .*

*Proof.* Let  $\Sigma_a(A) \neq 0$ . Suppose, to the contrary, that  $J \neq \Sigma_a(A)$ . If  $J/\Sigma_a(A)$  is nilpotent then there exists an ideal  $K$  of  $A$  such that  $\Sigma_a(A) \subsetneq K \subset J$  and  $K^2 \subset \Sigma_a(A)$ . Therefore  $[K, K] \subset \Sigma_a(A)$ . This is also true in the case when  $J/\Sigma_a(A)$  is commutative: in this case we can take  $K = J$ .

Let  $x, y \in K$  be arbitrary. Then  $z := [x, y] \in \Sigma_a(A)$ . By Lemma 5.1,

$$z^2 = z[x, y] = 0.$$

If  $z \neq 0$  then  $\mathfrak{P}_\beta(\Sigma_a(A)) \neq 0$ , whence  $\mathfrak{P}_\beta(A) \neq 0$ , a contradiction to the assumption that  $A$  is semiprime. Therefore  $z = 0$ , whence  $K$  is a commutative ideal of  $A$ . By definition,  $K \subset \Sigma_a(A)$ , a contradiction. Therefore  $J = \Sigma_a(A)$ .  $\square$

**Theorem 5.7.** *Let  $P \geq \mathfrak{P}_\beta$  be a weakly hereditary under radical. Then if one of the following conditions holds:*

- (1)  $\mathfrak{U}$  is [algebraically] universal and  $P$  is pliant or strict;
- (2)  $P$  is defined on Banach algebras and satisfies the condition of Banach heredity.

*then  $P^{*a}$  is a radical.*

*Proof.* First we prove the following statement:

(\*) *Let  $A$  be a  $P^*$ -semisimple algebra and  $B = A/\Sigma_a(A)$ . If  $B$  is commutative modulo  $P(B)$  then  $A = \Sigma_a(A)$ .*

Indeed,  $A$  is  $P$ -semisimple and semiprime. Let  $J := \Sigma_a(A)$ ,  $I = J^\perp$  (the left annihilator of  $J$ ). Then  $[A, A] \subset I$  by Lemma 5.1, hence either  $A$  is commutative or  $I \neq 0$ .

So we assume, to the contrary, that  $I \neq 0$ . If  $I \cap J \neq 0$  then it is a nilpotent ideal of  $A$ , a contradiction to semiprimeness of  $A$ .

Therefore  $I \cap J = 0$ , and the restriction  $f$  of  $q_J : A \rightarrow B$  to  $I$  is one-to-one onto  $K := q_J(I) \cong (I + J)/J$ . If  $K$  is commutative then  $I + J$  is commutative modulo  $J$ , whence  $I + J$  is commutative by Lemma 5.6, because  $A$  is semiprime. This contradicts to the maximality of  $\Sigma_a(A)$ . So  $K$  is not commutative. Since  $[K, K] \subset P(B)$ , we see that

$$K \cap P(B) \neq 0.$$

In the case when  $A$  is a Banach algebra,  $K$  is a Banach ideal of  $B$  (see Section 1.2.7). If  $P$  satisfies the condition of Banach heredity then

$$P(K; \|\cdot\|_K) = K \cap P(B) \neq 0.$$

Taking the norm  $\|\cdot\|_K$  as in Section 1.2.7, we have that  $f$  is an isometric isomorphism between  $I$  and  $(K; \|\cdot\|_K)$ . Then  $P(I) \neq 0$  whence  $P(A) \neq 0$ , a contradiction.

Let  $P$  be pliant or strict, and let  $L = K \cap P(B)$ . As  $L$  is a non-zero ideal of the  $P$ -radical algebra  $P(B)$  then

$$P(L) \neq 0$$

by the weak heredity of  $P$ .

Since  $f$  is a [bounded] isomorphism  $I \rightarrow K$ , then  $f(P(f^{-1}(L))) = P(L)$  whenever  $P$  is pliant, or  $\overline{f(P(f^{-1}(L)))} = P(L)$  whenever  $P$  is strict. In any case

$$P(f^{-1}(L)) \neq 0.$$

As  $f^{-1}(L)$  is an ideal of  $I$ , this implies that

$$0 \neq P(f^{-1}(L)) \subset P(I) \subset P(A),$$

a contradiction. The proof of the statement (\*) is completed.

Let now  $B$  be commutative modulo  $P^*(B)$ , and let  $(P_\alpha)$  be the convolution chain of  $P$ ; then  $P_1 = P$ . Let  $A_1 = \{x \in A : x/J \in P(B)\}$ . Then  $A_1$  is an ideal of  $A$ ; therefore  $A_1$  is  $P^*$ -semisimple, and  $J \subset \Sigma_a(A_1) = J \cap A_1$  by Lemma 5.1. So

$$\Sigma_a(A_1) = J.$$

Let  $B_1 = A_1/\Sigma_a(A_1)$ . As  $B_1 = P(B)$  then  $B_1 = P(B_1)$ , in particular  $B_1$  is commutative modulo  $P(B_1)$ . Then  $A_1$  is commutative by (\*). Therefore  $A_1 = J$  and  $P(B) = 0$ . Then  $P^*(B) = 0$ ,  $B$  is commutative and  $A$  is commutative by Lemma 5.6. As  $P^*$  is a radical,  $P^{*a}$  is a radical by Proposition 5.5.  $\square$

**Remark 5.8.** *In the assumptions of Theorem 5.7, if  $P^*$  is hereditary then  $P^{*a}$  is hereditary by Theorem 5.3. If  $P$  is an algebraic hereditary radical and  $P \geq \mathfrak{P}_\beta$ , then  $\overline{P}$  is a strict and weakly hereditary under radical by Theorem 3.5 and Lemma 4.26, whence  $\overline{P}^{*a}$  is a radical.*

**5.2. Centralization of classical radicals.** Now we apply Proposition 5.7 and Remark 5.8 to the classical radicals.

**Corollary 5.9.**  *$\text{rad}^a$ ,  $\overline{\text{rad}}^{*a}$ , and  $\text{Rad}^a$  are hereditary radicals.*

*Proof.* The radical  $\text{Rad}$  satisfies the condition of Banach heredity,  $\text{rad}$  is pliant and  $\overline{\text{rad}}$  is weakly hereditary. By Theorem 5.7, the centralizations of them are radicals. As  $\overline{\text{rad}}^*$  is hereditary,  $\overline{\text{rad}}^{*a}$  is a hereditary radical.  $\square$

**Theorem 5.10.**  $\text{Rad}^a(A) = \text{rad}^{\dim>1}(A) = \{x \in A : \rho(x[a, b]) = 0 \forall a, b \in A\}$  for every Banach algebra  $A$ .

*Proof.* Set  $J = \text{Rad}^a(A)$ . As  $\text{rad}^{\dim>1}(A)$  admits only one-dimensional strictly irreducible representations,  $\text{rad}^{\dim>1}(A) \subset J$ . If  $\pi(J) \neq 0$  for some  $\pi \in \text{Irr}^{\dim>1}(A)$  then  $\dim \pi(A) = \dim \pi(J) = 1$  by Jacobson's density theorem and commutativity of  $J$  modulo  $\text{Rad}(A)$ , a contradiction. So  $J \subset \text{rad}^{\dim>1}(A)$ .

Set  $I = \{x \in A : \rho(x[a, b]) = 0 \forall a, b \in A\}$ . Then  $J \subset I$  by (5.1) for  $P = \text{Rad}$ .

Let  $x \in I$  be arbitrary. Assume that  $\pi(x)\xi = \zeta \neq 0$  for some  $\pi \in \text{Irr}^{\dim>1}(A)$ . Taking a vector  $\eta$  such that  $\zeta$  and  $\eta$  are linearly independent, one can find  $a, b \in A$  by Jacobson's density theorem such that  $\pi(a)\eta = \xi$ ,  $\pi(a)\zeta = 0$  and  $\pi(b)\zeta = \eta$ . Then  $\pi([a, b])\zeta = \xi$ , whence

$$\pi(x[a, b])\zeta = \zeta,$$

a contradiction with  $\rho(x[a, b]) = 0$ . Therefore  $\pi(x) = 0$  for every  $\pi \in \text{Irr}^{\dim>1}(A)$ . This means that  $x \in \text{rad}^{\dim>1}(A)$  and  $I \subset J$ .  $\square$

**Remark 5.11.**  $\text{Rad}^{ar}(A) = \text{Rad}^{ra}(A) = \{x \in A : \rho(x[a, b]) = 0 \forall a, b \in \hat{A}\}$  for every normed algebra  $A$ .

Now we turn to the radicals of Baer, Levitzki, Köthe, to the hypofinite radical, and to their topological counterparts.

**Corollary 5.12.**  $\mathfrak{P}_\beta^a, \mathfrak{P}_\lambda^a, \mathfrak{P}_\kappa^a, \mathfrak{R}_{\text{hf}}^a, \mathcal{P}_\beta^a, \mathcal{P}_\lambda^a, \mathcal{P}_\kappa^a$  and  $\mathcal{R}_{\text{hf}}^a$  are hereditary radicals.

*Proof.* Follows from Theorem 5.7 and Remark 5.8 because all radicals are hereditary, and the closures of  $\mathfrak{P}_\beta, \mathfrak{P}_\lambda, \mathfrak{P}_\kappa, \mathfrak{R}_{\text{hf}}$  are strict and weakly hereditary.  $\square$

Recall that the closed-nil radical  $\mathcal{P}_{\text{nil}}$  is the restriction of  $\mathcal{P}_\beta$  to Banach algebras.

**Corollary 5.13.**  $\mathcal{P}_{\text{nil}}^a$  is a hereditary radical.

**Lemma 5.14.** Let  $P_1$  and  $P_2$  be radicals such that  $\mathfrak{P}_\beta \leq P_1 < P_2$ . If  $P_1 = P_2$  on commutative algebras then  $P_1^a < P_2^a$ .

*Proof.* Let  $A$  be an algebra such that  $P_1(A) \neq P_2(A)$ . Assume, to the contrary, that  $P_1^a(A) = P_2^a(A)$ . Let  $B = A/P_1(A)$ . Then  $B$  is semiprime and therefore commutative by Lemma 5.1. On the other hand,  $B$  is  $P_1$ -semisimple and therefore  $P_2$ -semisimple by our assumption. But  $B$  is not  $P_2$ -semisimple because  $P_2(A)/P_1(A)$  is  $P_2$ -radical, non-zero and lies in  $P_2(B)$ , a contradiction.  $\square$

**Corollary 5.15.**  $\text{rad}_n^a < \text{rad}_b^a < \text{Rad}^{ra}, \mathfrak{P}_\beta^a < \mathfrak{P}_\lambda^a < \mathfrak{P}_\kappa^a$  and  $\mathcal{P}_\beta^a < \mathcal{P}_\lambda^a < \mathcal{P}_\kappa^a$ .

*Proof.* The result follows from Lemma 5.14, (3.10), (3.8), Theorem 3.8, the coincidence of  $\text{Rad}^{ra}$  with  $\text{rad}_n^a$  on commutative algebras (by Theorem 5.10), and also the coincidence of  $\mathfrak{P}_\kappa^a$  with  $\mathfrak{P}_\beta^a$  on commutative algebras.  $\square$

**Problem 5.16.** Is  $\mathcal{R}_{\text{hc}}^a$  a radical?

**5.3. The centralization of the tensor Jacobson radical.** It is not clear if  $\mathcal{R}_{\text{cq}}$  and  $\mathcal{R}_t$  are strict, so we cannot apply Theorem 5.7 to them. However,  $\mathcal{R}_{\text{cq}}^a$  is a radical by [ST7], and it will be proved that  $\mathcal{R}_t^a$  is a radical.

First we note that the maps  $\mathcal{R}_t$  and  $\mathcal{R}_t^a$  are regular.

**Theorem 5.17.** Let  $A$  be a normed algebra. Then  $\mathcal{R}_t(B) = B \cap \mathcal{R}_t(A)$  and  $\mathcal{R}_t^a(B) = B \cap \mathcal{R}_t^a(A)$  for every dense subalgebra  $B$  of  $A$ .

*Proof.* As  $\mathcal{R}_t(B)$  is the set of all  $a \in B$  such that  $a \otimes c \in \text{Rad}(B \widehat{\otimes} C) = \text{Rad}(A \widehat{\otimes} C)$  for every normed algebra  $C$  and every  $c \in C$  [ST6], then clearly

$$\mathcal{R}_t(B) = B \cap \mathcal{R}_t(A).$$

By (5.1),  $\mathcal{R}_t^a(B) = \{x \in B : x[a, b] \in \mathcal{R}_t(B) \ \forall a, b \in B\}$ , whence we obtain that

$$\mathcal{R}_t^a(B) = B \cap \mathcal{R}_t^a(A).$$

□

We need here and will use also in the further sections a result on tensor radius of operator families, analogous to the result about the joint spectral radius established in [ST1, Lemma 4.2]. If  $T$  is an operator on a space  $X$  and  $Y$  is an invariant subspace for  $T$  then  $T|_Y$  denotes the restriction of  $T$  to  $Y$ , and  $T|_{X/Y}$  — the operator induced by  $T$  on  $X/Y$ . If  $Y$  is invariant for a set  $M$  of operators then  $M|_Y = \{T|_Y : T \in M\}$  and  $M|_{X/Y} = \{T|_{X/Y} : T \in M\}$ ; this is transferred similarly to families of operators.

Recall that the tensor spectral radius  $\rho_t$  is defined in Section 2.3.3 and

$$\rho_t(M^m) = \rho_t(M)^m$$

for every  $m > 0$  and every summable family  $M$  in a normed algebra.

**Theorem 5.18.** *Let  $M = (a_i)_1^\infty$  be a summable family of operators in  $\mathcal{B}(X)$ , and let  $Y$  be an invariant closed subspace for  $M$ . Then*

$$\rho_t(M) = \max \{ \rho_t(M|_{X/Y}), \rho_t(M|_Y) \}.$$

*Proof.* It is clear that  $\max \{ \rho_t(M|_{X/Y}), \rho_t(M|_Y) \} \leq \rho_t(M)$ . Let us prove the converse inequality. Let  $q : X \rightarrow V := X/Y$  be the standard quotient map. Note that

$$\|M^{2n}\|_+ = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \|b_i b_j\|$$

where  $M^n = (b_i)_1^\infty$ . For every  $\varepsilon > 0$ , there is  $x_{ij} \in X$  with  $\|x_{ij}\| = 1$  such that  $\|b_i b_j\| \leq \|b_i b_j x_{ij}\| + \varepsilon$ . Clearly  $\|b_i b_j x_{ij}\| \leq \|b_i y\| + \|b_i\| \|b_j x_{ij} - y\|$  for any  $y \in Y$ . Take  $y = y_{ij}$  such that  $\|b_j x_{ij} - y_{ij}\| \leq \|q(b_j x_{ij})\| + \varepsilon$ . Then

$$\|y_{ij}\| \leq \|b_j x_{ij}\| + \|b_j x_{ij} - y_{ij}\| \leq \|b_j x_{ij}\| + \|q(b_j x_{ij})\| + \varepsilon.$$

Hence

$$\begin{aligned} \|b_i b_j\| &\leq \|b_i b_j x_{ij}\| + \varepsilon \leq \|b_i y_{ij}\| + \|b_i\| \|b_j x_{ij} - y_{ij}\| + \varepsilon \\ &\leq \|b_i|_Y\| (\|b_j x_{ij}\| + \|q(b_j x_{ij})\| + \varepsilon) + \|b_i\| (\|q(b_j x_{ij})\| + \varepsilon) + \varepsilon \end{aligned}$$

Taking the limit for  $\varepsilon \rightarrow 0$ , we get

$$\|b_i b_j\| \leq \|b_i|_Y\| (\|b_j\| + \|b_j|_V\|) + \|b_i\| \|b_j|_V\|.$$

Therefore

$$\begin{aligned} \|M^{2n}\|_+ &\leq \sum_i \sum_j (\|b_i|_Y\| \|b_j\| + \|b_i|_Y\| \|b_j|_V\| + \|b_i\| \|b_j|_V\|) \\ &\leq \|M^n|_Y\|_+ \|M^n\|_+ + \|M^n|_Y\|_+ \|M^n|_V\|_+ + \|M^n\|_+ \|M^n|_V\|_+. \end{aligned}$$

Let  $n = 2k$  in this inequality; as  $\|M^n\|_+ \leq \|M^k\|_+^2$  and

$$\begin{aligned} \|M^n|_Y\|_+ &\leq \|M^k|_Y\|_+ \|M^k|_Y\|_+ \leq \|M^k|_Y\|_+ \|M^k\|_+, \\ \|M^n|_V\|_+ &\leq \|M^k|_V\|_+ \|M^k|_V\|_+ \leq \|M^k|_V\|_+ \|M^k\|_+ \end{aligned}$$

then

$$\begin{aligned} \|M^{2n}\|_+ &\leq \|M^k\|_+^2 \left( \|M^k|_Y\|_+^2 + \|M^k|_Y\| \|M^k|_V\| + \|M^k|_V\|_+^2 \right) \\ &\leq \|M^k\|_+^2 \left( \|M^k|_Y\|_+ + \|M^k|_V\|_+ \right)^2 \\ &\leq 4 \|M^k\|_+^2 \max \left\{ \|M^k|_Y\|_+, \|M^k|_V\|_+ \right\}^2 \end{aligned}$$

Taking roots and setting  $k \rightarrow \infty$ , we obtain

$$\rho_t(M) = \rho_t(M)^{1/2} \max \{ \rho_t(M|_Y), \rho_t(M|_V) \}^{1/2},$$

i.e.,  $\rho_t(M) \leq \max \{ \rho_t(M|_Y), \rho_t(M|_V) \}$ .  $\square$

Let  $A$  be a normed algebra. Recall that, for every element  $a \in A$ ,  $L_a$  and  $R_a$  are defined as operators  $x \mapsto ax$  and  $x \mapsto xa$  on  $A$ , respectively. If  $M = (a_n)_1^\infty$  is a family in  $A$ , let  $L_M$  and  $R_M$  denote the operator families  $(L_{a_n})_1^\infty$  and  $(R_{a_n})_1^\infty$ , respectively.

**Theorem 5.19.** *Let  $A$  be a normed algebra, and let  $M = (a_n)_1^\infty$  be a summable family in  $A$ . Then  $\rho_t(M) = \rho_t(L_M) = \rho_t(R_M)$  and  $\rho_t(L_MR_M) = \rho_t(M)^2$ .*

*Proof.* It is clear that  $\rho_t(L_M) \leq \rho_t(M)$ , and  $\rho_t(L_MR_M) \leq \rho_t(L_M) \rho_t(R_M) \leq \rho_t(M)^2$  by [ST6, Proposition 3.4]. Further,

$$\begin{aligned} \|M^{m+1}\|_+ &= \|L_M^m(M)\|_+ = \sum_{i_1=1}^\infty \cdots \sum_{i_n=1}^\infty \sum_{i_{n+1}=1}^\infty \|L_{a_{i_1}} \cdots L_{a_{i_n}} a_{i_{n+1}}\| \\ &\leq \sum_{i_1=1}^\infty \cdots \sum_{i_n=1}^\infty \sum_{i_{n+1}=1}^\infty \|L_{a_{i_1}} \cdots L_{a_{i_n}}\| \|a_{i_{n+1}}\| = \|L_M^m\|_+ \|M\|_+ \end{aligned}$$

for every  $m > 0$ ; this implies  $\rho_t(M) \leq \rho_t(L_M)$ . Similarly,

$$\|M^{2m+1}\|_+ \leq \|(L_MR_M)^m\|_+ \|M\|_+,$$

for every  $m > 0$ , implies  $\rho_t(M)^2 = \rho_t(M^2) \leq \rho_t(L_MR_M)$ .  $\square$

Recall that  $M/I$  is the image of  $M$  in  $A/I$ .

**Corollary 5.20.** *Let  $A$  be a normed algebra, and let  $I$  be a closed ideal of  $A$ . Then  $\rho_t(M) = \max \{ \rho_t(M/I), \rho_t(L_M|_I) \}$  for every summable family  $M = (a_n)_1^\infty$  in  $A$ .*

*Proof.* Indeed, it follows from Theorems 5.19 and 5.18 that

$$\begin{aligned} \rho_t(M) &= \rho_t(L_M) = \max \{ \rho_t(L_M|_{A/I}), \rho_t(L_M|_I) \} \\ &= \max \{ \rho_t(L_{M/I}), \rho_t(L_M|_I) \} = \max \{ \rho_t(M/I), \rho_t(L_M|_I) \}. \end{aligned}$$

$\square$

For a summable family  $M = (a_n)_1^\infty$  in  $A$ , let

$$M|_k = (a_n)_1^k, \quad M|_{k+1} = (a_n)_{k+1}^\infty \quad \text{and} \quad \rho_+(M) = \sum_1^\infty \rho(a_n).$$

As  $\rho(a_n) \leq \|a_n\|$  for every  $n > 0$ , then  $\rho_+(M) < \infty$  for every summable family  $M$ .

**Remark 5.21.** *The inequality  $\rho_+(M) \leq \rho_t(M)$  does not hold even for commutative Banach algebra. Indeed, if  $M = (p, 1-p, 0, \dots)$  for a non-trivial idempotent  $p$  then  $\rho_t(M) = 1$  while  $\rho_+(M) = 2$ . See the calculation of  $\rho_t(M)$  for commutative finite families  $M$  via joint spectra in [M].*

**Lemma 5.22.** *Let  $M = (a_n)_1^\infty$  be a commutative summable family in  $A$ . Then  $\rho_t(M) \leq \rho_+(M)$ .*

*Proof.* As  $M|_k$  and  $M|_{k+1}$  commute for every  $k > 0$ , and  $M = M|_k \sqcup M|_{k+1}$ , we obtain that

$$\rho_t(M) \leq \rho_t(M|_k) + \rho_t(M|_{k+1}) \leq \rho_+(M|_k) + \rho_t(M|_{k+1})$$

by [ST6, Proposition 3.4]. For every  $\varepsilon > 0$  there is  $n > 0$  such that  $\rho_t(M|_{n+1}) \leq \|M|_{n+1}\|_+ < \varepsilon$ , whence

$$\rho_t(M) \leq \sup_k \rho_t(M|_k) = \sup_k \rho_+(M|_k) = \rho_+(M).$$

□

**Corollary 5.23.** *Let  $A$  be a normed algebra,  $I$  a closed ideal of  $A$ , and let  $M = (a_n)_1^\infty$  be a summable family in  $A$ . Then*

- (1) *If  $A/I$  is commutative then  $\rho_t(M) \leq \max\{\rho_+(M), \rho_t(L_M|_I)\}$ ;*
- (2) *If  $I$  is central then, for every  $k > 0$ ,*

$$\rho_t(M) \leq \max\{\rho_t(M/I), \rho_t(M|_k) + \rho_t(M|_{k+1})\}.$$

*Proof.* (1) It follows from Lemma 5.22 and Corollary 5.20 that

$$\rho_t(M) \leq \max\{\rho_+(M/I), \rho_t(L_M|_I)\} \leq \max\{\rho_+(M), \rho_t(L_M|_I)\}$$

since  $\rho(a_n/I) \leq \rho(a_n)$  for every  $n$ .

(2) It is easy to check that  $L_M|_I$  is commutative. Then

$$\begin{aligned} \rho_t(L_M|_I) &\leq \rho_t(L_{M|_k}|_I) + \rho_t(L_{M|_{k+1}}|_I) \leq \rho_t(M|_k/I) + \rho_t(M|_{k+1}/I) \\ &\leq \rho_t(M|_k) + \rho_t(M|_{k+1}) \end{aligned}$$

by Lemma 5.22, and

$$\begin{aligned} \rho_t(M) &= \max\{\rho_t(M/I), \rho_t(L_M|_I)\} \\ &\leq \max\{\rho_t(M/I), \rho_t(M|_k) + \rho_t(M|_{k+1})\}. \end{aligned}$$

□

**Corollary 5.24.** *Let  $A$  be a normed algebra. If  $A/\mathcal{R}_t(A)$  is commutative then  $\rho_t(M) \leq \rho_t(M|_k) + \rho_t(M|_{k+1})$  for every summable family  $M = (a_n)_1^\infty$  in  $A$  and every  $k > 0$ .*

*Proof.* By (2.5) and Lemma 5.22,

$$\rho_t(M) = \rho_t(M/\mathcal{R}_t(A)) \leq \rho_t(M|_k) + \rho_t(M|_{k+1}).$$

□

**Lemma 5.25.** *Let  $A$  be an  $\mathcal{R}_t$ -semisimple algebra, and let  $I$  be a closed commutative ideal of  $A$ . If  $A/I$  is commutative modulo  $\mathcal{R}_t(A/I)$  then  $\rho_t(\{a\} \sqcup M) \leq \rho(a) + \rho_t(M)$  for every  $a \in A$  and every summable family  $M = (a_n)_1^\infty$  in  $A$ .*

*Proof.* As  $A$  is a semiprime algebra,  $I$  is a central ideal of  $A$  by Lemma 5.1. By Corollary 5.23,

$$\rho_t(\{a\} \sqcup M) \leq \max\{\rho_t((\{a\} \sqcup M)/I), \rho(a) + \rho_t(M)\}.$$

By Corollary 5.24,

$$\rho_t((\{a\} \sqcup M)/I) \leq \rho(a/I) + \rho_t(M/I) \leq \rho(a) + \rho_t(M).$$

Therefore  $\rho_t(\{a\} \sqcup M) \leq \rho(a) + \rho_t(M)$ . □

In the following theorem we will use the inequality

$$\rho_t(\{a+b\} \sqcup M) \leq \rho_t(\{a\} \sqcup \{b\} \sqcup M) \quad (5.3)$$

for every  $a, b \in A$  and every summable family  $M$  in  $A$ ; this is a special case of [ST6, Proposition 3.3].



**Theorem 5.26.**  $\mathcal{R}_t^a$  is a uniform regular topological radical on  $\mathfrak{U}_n$ .

*Proof.* Let  $A$  be an  $\mathcal{R}_t$ -semisimple algebra, and let  $I$  be a closed commutative ideal of  $A$  such that  $A/I$  is commutative modulo  $\mathcal{R}_t(A/I)$ . By Proposition 5.5, to prove that  $\mathcal{R}_t^a$  is a topological radical, it suffices to show that  $A$  is commutative.

Assume that  $A$  is a Banach algebra. Let  $M = (a_n)_1^\infty$  be a summable family in  $A$ , and  $a, b \in A$  be arbitrary. Put  $c_\lambda = \exp(\lambda \operatorname{ad}(b))(a)$  for every  $\lambda \in \mathbb{C}$ , and put  $f(\lambda) = (c_\lambda - a)/\lambda$ ; then  $\lambda \mapsto f(\lambda)$  is an analytic function in  $A$  and  $f(0) = [b, a]$ . As  $\rho(c_\lambda) = \rho(a)$  then, by (5.3) and Lemma 5.25,

$$\rho_t(\{c_\lambda - a\} \sqcup M) \leq \rho_t(\{c_\lambda\} \sqcup \{a\} \sqcup M) \leq 2\rho(a) + \rho_t(M)$$

for every  $\lambda \in \mathbb{C}$ . Replace  $a$  by  $a/\lambda$  for  $\lambda \neq 0$ ; we obtain that

$$\rho_t(\{f(\lambda)\} \sqcup M) \leq 2\rho(a)/\lambda + \rho_t(M) \quad (5.4)$$

The function  $\lambda \mapsto \rho_t(\{f(\lambda)\} \sqcup M)$  is subharmonic by [ST6, Theorem 3.16] and bounded on  $\mathbb{C}$  by (5.4); therefore it is constant and

$$\rho_t(\{[b, a]\} \sqcup M) = \rho_t(\{f(0)\} \sqcup M) = \lim_{|\lambda| \rightarrow \infty} \rho_t(\{f(\lambda)\} \sqcup M) \leq \rho_t(M).$$

As  $M$  is an arbitrary summable family in  $A$  then  $[b, a] \in \mathcal{R}_t(A) = 0$ , therefore  $A$  is commutative.

As  $\mathcal{R}_t$  is hereditary, it follows from Theorem 5.3 and Proposition 5.5 that  $\mathcal{R}_t^a$  is a hereditary topological radical on  $\mathfrak{U}_b$ . As  $\mathcal{R}_t^a$  is regular by Theorem 5.17,  $\mathcal{R}_t^a$  is a hereditary topological radical on  $\mathfrak{U}_n$ .

Let  $A$  be an  $\mathcal{R}_t^a$ -radical algebra, and let  $B$  be a subalgebra of  $A$ . If  $a, b \in B$  then  $\rho_t(\{[b, a]\} \sqcup M) = \rho_t(M)$  for every summable family in  $B$ . Therefore  $B$  is  $\mathcal{R}_t^a$ -radical and  $\mathcal{R}_t^a$  is uniform.  $\square$

**5.4. Spectral applications.** Let  $\mathfrak{U}$  be a class of algebras. A property of algebras from  $\mathfrak{U}$  is called *radical* (respectively, *semisimple*) in  $\mathfrak{U}$  if algebras with this property form the radical (respectively, semisimple) class for some radical.

We start with  $\operatorname{Rad}^a$ -radical algebras. It is well known that Banach algebras commutative modulo the Jacobson radical share with the commutative ones the advantages of easy calculation of spectra and the continuity of the functions  $a \rightarrow \sigma(a)$  and  $a \rightarrow \rho(a)$ . Moreover, it was shown in [PZ, Z1] that a Banach algebra is commutative modulo radical if and only if one (or all) of the following conditions holds:

- (1<sub>a</sub>) The function  $a \rightarrow \rho(a)$  is submultiplicative;
- (2<sub>a</sub>) The function  $a \rightarrow \rho(a)$  is subadditive;
- (3<sub>a</sub>) The function  $a \rightarrow \rho(a)$  is uniformly continuous.

Using the fact that  $\operatorname{Rad}^a$  is a radical (Corollary 5.9), we obtain the following result:

**Corollary 5.27.** *Let  $A$  be a Banach algebra, let  $I$  be a closed ideal of  $A$  and let  $\mathcal{F}$  be a family of closed ideals of  $A$  with dense sum in  $A$ . If either  $I$  and  $A/I$ , or all ideals from  $\mathcal{F}$  have one of the properties (1<sub>a</sub>), (2<sub>a</sub>), (3<sub>a</sub>) then  $A$  has the same property.*

In other words, the uniform continuity of  $a \rightarrow \rho(a)$  is a radical property of Banach algebras.

**Problem 5.28.** *Will the statement of Corollary 5.27 stay true if one replaces the uniform continuity of  $a \rightarrow \rho(a)$  by the continuity of this map?*

Some results related to this problem can be found in Section 9.

In the theory of joint spectra and spectral radii the commutativity modulo the radical  $\mathcal{R}_{\text{cq}}$  or  $\mathcal{R}_t$  plays a role which is at least partially similar to the role of commutativity modulo  $\text{Rad}$  in the “individual” spectral theory.

**Theorem 5.29.** *Let  $A$  be a normed algebra. Then*

- (1) *If  $M = \{a_\alpha : \alpha \in \Lambda\}$  is a family in  $A$  which is precompact and commutative modulo  $\mathcal{R}_{\text{cq}}(A)$  then  $\rho(M) = \sup \{|\lambda| : \lambda \in \sigma_A^l(M)\}$ ; (Here  $|\lambda| = \sup_\alpha |\lambda_\alpha|$ .)*
- (2) *If  $M = (a_n)_1^\infty$  is a summable family which is commutative modulo  $\mathcal{R}_t(A)$  then  $\rho_t(M) = \sup \{|\lambda|_+ : \lambda \in \sigma_A^l(M)\} = \sup \{|\lambda|_+ : \lambda \in \sigma_A^r(M)\}$ . (Here  $|\lambda|_+ = \sum_1^\infty |\lambda_n|$ .)*

*Proof.* As  $\mathcal{R}_{\text{cq}}(A) = A \cap \mathcal{R}_{\text{cq}}(\widehat{A})$ ,  $\mathcal{R}_t(A) = A \cap \mathcal{R}_t(\widehat{A})$  and the values of the joint spectral and tensor radii do not change under taking the completion of  $A$ , one may assume that  $A$  is a Banach algebra.

(1) As  $\rho(M) = \rho(M/\mathcal{R}_{\text{cq}}(A))$  by [ST6, Theorem 3.29], one may assume that  $M$  is commutative. By [ST7, Theorem 6.3], the joint spectral radius is continuous at  $M$  and

$$\rho(M) = \sup \{\rho(N) : N \text{ is a finite subfamily of } M\}. \quad (5.5)$$

The first equality below holds by [M, Theorem 35.5] while the other relations are obvious:

$$\begin{aligned} \rho(N) &= \sup \{|\lambda| : \lambda \in \sigma_A^l(N)\} \\ &\leq \sup \{|\lambda| : \lambda \in \sigma_A^l(M)\} = \sup_{a_\alpha \in M} \rho(a_\alpha) \leq \rho(M). \end{aligned} \quad (5.6)$$

The result follows from (5.5) and (5.6).

(2) As  $\rho(M) = \rho(M/\mathcal{R}_t(A))$  by [ST5, Theorem 4.18], one may assume that  $M$  is commutative. As  $M$  is summable then for every  $\varepsilon > 0$  there is  $n > 0$  such that  $\sum_{n+1}^\infty \|a_k\| < \varepsilon$ . By Corollary 5.24,

$$\rho(M|^\infty) \leq \rho(M) \leq \rho(M|^\infty) + \rho(M|_{n+1}) \leq \rho(M|^\infty) + \sum_{n+1}^\infty \|a_k\| < \rho(M|^\infty) + \varepsilon$$

whence  $\rho(M) = \lim_{n \rightarrow \infty} \rho(M|^\infty)$ . By [M, Theorem 35.6],

$$\rho(M|^\infty) = \sup \{|\lambda|_+ : \lambda \in \sigma_A^l(M|^\infty)\},$$

whence the result follows.  $\square$

It should be noted that the results of Müller, used in the above proof, were formulated in [M] for finite families in Banach algebras and for the Harte spectrum, but they hold also for the left and right spectra due to [M, Proposition 35.2].

It follows from Theorem 5.29(1) that  $\rho(M) = \sup_{a_\alpha \in M} \rho(a_\alpha)$  under the posed conditions (this was proved also in [ST7]). As a consequence, we obtain that each  $\mathcal{R}_{\text{cq}}^a$ -radical algebra is a Berger-Wang algebra (i.e.  $\rho(M) = r(M)$  for all precompact sets  $M$  in  $A$ ). By [ST7, Corollary 5.15],  $\mathcal{R}_{\text{hc}} \vee \mathcal{R}_{\text{cq}}^a \leq \mathcal{R}_{\text{bw}}$  and in particular for every normed algebra  $A$  the following *algebra version of the joint spectral radius formula*

$$\rho(M) = \max \{\rho(M/(\mathcal{R}_{\text{hc}} \vee \mathcal{R}_{\text{cq}}^a)(A)), r(M)\}$$

holds for every precompact set  $M$  in  $A$ . We do not know however whether every Berger-Wang algebra is  $\mathcal{R}_{\text{bw}}$ -radical.

**Proposition 5.30.** *Let  $A$  be a normed algebra. If  $A/\mathcal{R}_{\text{bw}}(A)$  is a Berger-Wang algebra then  $A$  is a Berger-Wang algebra.*

*Proof.* Indeed,

$$\rho(M) = \max\{\rho(M/\mathcal{R}_{\text{bw}}(A)), r(M)\} = \max\{r(M/\mathcal{R}_{\text{bw}}(A)), r(M)\} = r(M)$$

for every precompact set  $M$  in  $A$ . Therefore  $A$  is a Berger-Wang algebra.  $\square$

The following theorem supplies us with a class of examples of Banach algebras which are  $\mathcal{R}_{\text{cq}}^a$ -radical (and therefore  $\mathcal{R}_t^a$ -radical); these algebras hold an important place in the theory of joint spectra of Lie representations (see for instance [F, BS, Ds]).

Recall that algebras can be considered as Lie algebras with respect to the Lie bracket  $[a, b] = ab - ba$ . A Lie subalgebra of an algebra  $A$  is a subspace  $L \subset A$  with the property that  $[a, b] \in L$  for all  $a, b \in L$ .

Operators  $\text{ad}_L(a) : x \mapsto [a, x]$  on a Lie algebra  $L$ , for  $a \in L$ , are called *adjoint operators* of  $L$ .

For a Lie algebra  $L$ , one defines the upper (lower) central series  $L^{[n]}$  (respectively  $L_{[n]}$ ) by setting  $L_{[1]} = L^{[1]} = L$ ,  $L_{[n+1]} = [L, L_{[n]}]$  and  $L^{[n+1]} = [L^{[n]}, L^{[n]}]$ ;  $L$  is called *nilpotent* (*solvable*) if  $L_{[n]} = 0$  (respectively  $L^{[n]} = 0$ ), for some  $n$ .

**Theorem 5.31.** *Let  $A$  be a normed algebra and let  $L$  be a Lie subalgebra of  $A$ . If one of the following conditions holds:*

- (1)  *$L$  is a nilpotent Lie algebra, and the inverse-closed subalgebra generated by  $L$  is dense in  $A^1$ ;*
- (2)  *$L$  is a finite-dimensional solvable Lie algebra, and the subalgebra generated by  $L$  is dense in  $A$ ;*

*then  $A$  is commutative modulo  $\mathcal{R}_{\text{cq}}(A)$ , i.e.,  $A = \mathcal{R}_{\text{cq}}^a(A)$ .*

*Proof.* Let  $A' = A/\mathcal{R}_{\text{cq}}(A)$ , and let  $L'$  be the image of  $L$  in  $A'$ .

(1) If  $L'$  is not commutative then, by the Kleinecke-Shirokov theorem, there is a non-zero quasinilpotent element  $x \in [L', L']$  in the center of  $L'$ . Let  $B_0$  be the subalgebra of  $A'$  generated by  $L'$ . Then clearly  $x$  is in the center of  $B_0$ . Let  $B_1$  be the subalgebra generated by  $B_0$  and all inverses of elements from  $B_0$ . Then  $x$  is again in the center of  $B_1$ . If we have already built  $B_n$  and proved that  $x$  is in the center of  $B_n$ , then  $B_{n+1}$  is defined as a subalgebra generated by  $B_n$  and all inverses of elements from  $B_n$ , and it is clear that  $x$  is in the center of  $B_{n+1}$ .

Let  $C = \cup_{n=0}^{\infty} B_n$ . Then  $C$  is an inverse-closed subalgebra of  $A'$ ; indeed, if  $a \in B_n$  has an inverse  $a^{-1}$  in  $A'$  then  $a^{-1} \in B_{n+1}$ . It is clear that  $C$  is dense in  $A'$ . Thus  $x$  is in the center of  $A'$ . As  $x$  commutes with elements of each  $M \in \mathfrak{k}(A')$  then

$$\rho(\{x\} \cup M) \leq \max\{\rho(x), \rho(M)\} = \rho(M)$$

since  $x$  is a quasinilpotent element. Therefore  $x \in \mathcal{R}_{\text{cq}}(A') = 0$  (see Section 2.3.4), a contradiction.

So  $L'$  is commutative, whence  $A$  is commutative modulo  $\mathcal{R}_{\text{cq}}(A)$ .

(2) Let  $I$  be the set of nilpotents in  $L'$ . By [BS, Proposition 24.1],  $I$  is a Lie ideal of  $L'$ . As  $I$  is finite-dimensional, the nilpotency indexes of elements of  $I$  are uniformly bounded, so  $I$  is a nilpotent Lie algebra (by the algebraic Engel condition) and, moreover, generates nilpotent subalgebra in  $A$ , whence  $I^n = 0$  for some  $n > 0$ . As  $L'I \subset IL' + I$ , it follows that  $I$  generates a nilpotent ideal in the subalgebra generated by  $L'$  and generates therefore a closed nilpotent ideal in  $A'$ . But every nilpotent ideal lies in  $\mathcal{R}_{\text{cq}}(A') = \mathcal{R}_{\text{cq}}(A/\mathcal{R}_{\text{cq}}(A)) = 0$ , whence  $I = 0$ . This implies that  $L'$  is a nilpotent Lie algebra; indeed, every eigenvector  $x$  of an adjoint operator  $\text{ad}_{L'}(a)$  corresponding to a non-zero eigenvalue is nilpotent which is impossible. By (1),  $A' = \mathcal{R}_{\text{cq}}^a(A')$ , whence  $A'$  is commutative and then  $A = \mathcal{R}_{\text{cq}}^a(A)$ .  $\square$

Now we will discuss conditions under which the algebras generated (in the above sense) by nilpotent Lie subalgebras belong to a special subclass of the class of  $\text{Rad}^a$ -radical algebras; this subclass occupies an important place in the theory of linear operator equations and, more generally, in the study of multiplication operators on Banach algebras (see [ST6]).

A normed algebra  $A$  is called an *Engel algebra* if all operators  $\text{ad}_A(a)$ ,  $a \in A$ , are quasinilpotent. It was proved in [ST4] (and can be deduced from earlier results of [AM]) that each Engel Banach algebra  $A$  is commutative modulo  $\text{Rad}(A)$ . The converse is not true even for finite-dimensional algebras, for instance for the algebra of all upper-triangular matrices. Clearly the class of all Engel algebras is stable under taking closed ideals and quotients; it evidently contains all commutative and all radical Banach algebras; as a consequence, this class is not stable under extensions.

An operator  $T$  on an algebra  $A$  is called *elementary* if  $T = \sum_{i=1}^n L_{a_i} R_{b_i}$  for some  $a_1, b_1, \dots, a_n, b_n \in A^1$ . For instance, the identity operator on  $A$  is also elementary. Let  $\mathcal{E}\ell(A)$  be the algebra of all elementary operators on  $A$ .

**Lemma 5.32.** *Let  $A$  be a Banach algebra,  $L$  be a nilpotent Lie subalgebra of  $A$ . Then the closed, inverse-closed subalgebra  $B$  of  $\mathcal{B}(A^1)$  generated by  $L_L + R_L$  is commutative modulo  $\mathcal{R}_{\text{cq}}(B)$ .*

*Proof.* Since the closure of an inverse-closed subalgebra of a Banach algebra is inverse-closed and  $L_L + R_L$  is a nilpotent Lie algebra, we obtain that  $B$  is commutative modulo  $\mathcal{R}_{\text{cq}}(B)$  by Theorem 5.31.  $\square$

**Theorem 5.33.** *Let  $A$  be a Banach algebra, and let  $L$  be a nilpotent Lie subalgebra of  $A$  such that the inverse-closed subalgebra generated by  $L$  is dense in  $A^1$ . Then the following statements are equivalent:*

- (1)  $A$  is an Engel algebra;
- (2) For every elementary operator  $\sum L_{a_i} R_{b_i}$ ,

$$\sigma_{\mathcal{B}(A^1)} \left( \sum_1^n L_{a_i} R_{b_i} \right) = \sigma_{A^1} \left( \sum_1^n a_i b_i \right);$$

- (3) The closed subalgebra generated by  $\text{ad}_A(A)$  is compactly quasinilpotent;
- (4)  $\text{ad}_A(a)$  is quasinilpotent for every  $a \in L \setminus [L, L]$ .

*Proof.* (2)  $\Rightarrow$  (1)  $\Rightarrow$  (4) and (3)  $\Rightarrow$  (1) are obvious; (1)  $\Rightarrow$  (3) follows from Theorem 5.31 and Lemma 5.32.

(1)  $\Rightarrow$  (2) Let  $B$  be the closed, inverse-closed subalgebra of  $\mathcal{B}(A^1)$  generated by  $L_L + R_L$ , and let  $T = \sum_1^n L_{a_i} (R_{b_i} - L_{b_i})$  and  $c = \sum_1^n a_i b_i$ . As  $B$  is inverse-closed,

$$\sigma_{\mathcal{B}(A^1)} \left( \sum_1^n L_{a_i} R_{b_i} \right) = \sigma_B \left( \sum_1^n L_{a_i} R_{b_i} \right) = \sigma_B (T + L_c).$$

By Lemma 5.32,  $B$  is commutative modulo  $\mathcal{R}_{\text{cq}}(B)$ . As  $\mathcal{R}_{\text{cq}}(B) \subset \text{Rad}(B)$  then  $B$  is commutative modulo  $\text{Rad}(B)$ , whence the set of all quasinilpotent elements of  $B$  coincides with  $\text{Rad}(B)$ . Then, as  $A$  is Engel, it follows that  $T \in \text{Rad}(B)$  and  $\sigma_B(T + L_c) = \sigma_B(L_c)$ . But  $\sigma_B(L_c) = \sigma_{\mathcal{B}(A^1)}(L_c) = \sigma_{A^1}(c)$  by [BD, Proposition 3.19], and the result follows.

(4)  $\Rightarrow$  (1) As  $[L, L]$  consists of quasinilpotents, it follows that  $\text{ad}_A(L)$  consists of quasinilpotents. Let  $E = \{a \in A^1 : \rho(\text{ad}_{A^1}(a)) = 0\}$ . It is easy to see that  $E$  is an algebra by using Lemma 5.32. If  $x \in E$  is invertible in  $A$  then

$$\rho(\text{ad}_{A^1}(x^{-1})) = \rho(L_{x^{-1}} \text{ad}_{A^1}(x) R_{x^{-1}}) \leq \rho(L_{x^{-1}}) \rho(\text{ad}_{A^1}(x)) \rho(R_{x^{-1}}) = 0$$

by Lemma 5.32, whence  $E$  is an inverse-closed subalgebra of  $A$ . As  $L \subset E$  then  $E$  is dense in  $A$ . Let  $B$  be defined as in Lemma 5.32; as  $B$  is commutative modulo  $\text{Rad}(B)$  then the spectral radius is continuous on  $B$ . Then  $E$  is closed and  $A = E$ .  $\square$

**Corollary 5.34.** *Let  $A$  be a Banach algebra, and let  $L$  be a nilpotent Lie subalgebra of  $A$  such that the inverse-closed subalgebra generated by  $L$  is dense in  $A^1$ . If  $\sigma_{A^1}(a)$  is at most countable for every  $a \in L \setminus [L, L]$ , then  $A$  is an Engel algebra.*

*Proof.* Let  $A_0$  be the subalgebra of  $A^1$  generated by  $L$  and the identity element 1 of  $A^1$ . It follows that  $\sigma_{B(A^1)}(\text{ad}_{A^1}(a))$  is at most countable for every  $a \in L \setminus [L, L]$ . As  $\text{ad}_{A^1}(a)$  is nilpotent on  $L$ , it is locally nilpotent on  $A_0$  and is quasinilpotent on the closure  $B$  of  $A_0$  by [ST4, Corollary 3.7]. To apply Theorem 5.33, it is sufficient to show  $B$  is an inverse-closed subalgebra of  $A$ .

Let  $x \in B$  and  $x^{-1} \in A$ . Then there is a sequence  $x_n$  of elements of  $A_0$  such that  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . As the map  $a \mapsto a^{-1}$  is continuous on the set of all invertible elements of  $A$  and this set is open, one can assume that for every  $n$  there is  $x_n^{-1} \in A$  and the sequence  $x_n^{-1} \rightarrow x^{-1}$  as  $n \rightarrow \infty$ .

Note that, for every  $n$ , there is a polynomial  $p_n$  such that  $x_n = p_n(a_1, \dots, a_k)$  for some  $a_1, \dots, a_k \in L \setminus [L, L]$ . By the spectral mapping theorem [T1], we obtain that

$$\sigma_{A^1}(p_n(a_1, \dots, a_k)) = p_n(\sigma_{A^1}(a_1), \dots, \sigma_{A^1}(a_k)) \subset p_n(\sigma_{A^1}(a_1) \times \dots \times \sigma_{A^1}(a_k)).$$

As  $\sigma_{A^1}(a_1) \times \dots \times \sigma_{A^1}(a_k)$  is at most countable, it follows that the spectrum  $\sigma_{A^1}(x_n)$  is also at most countable. By [BD, Theorem 5.11],  $x_n^{-1}$  lies in the closed subalgebra of  $A^1$  generated by  $x_n$  and 1. As  $x_n, 1 \in B$  then  $x_n^{-1} \in B$  for every  $n$ . Therefore  $x^{-1} \in B$ .

We showed that  $B$  is inverse-closed. Therefore  $B = A^1$ . As  $\text{ad}_A(a)$  is quasinilpotent for every  $a \in L \setminus [L, L]$ , it follows that  $A$  is an Engel algebra by Theorem 5.33.  $\square$

Some related results are obtained in [C].

**Problem 5.35.** *Does every Banach algebra have the largest Engel ideal?*

## 6. SOCLE PROCEDURE AND RADICALS

**6.1. Socle.** Let  $A \in \mathfrak{U}_a$  be an algebra. The (left) *socle*  $\text{soc}(A)$  of  $A$  is the sum of all minimal left ideals of  $A$ . If  $A$  has no minimal left ideals then  $\text{soc}(A) = 0$ ;  $\text{soc}(A)$  is an ideal of  $A$  [BD, Lemma 30.9]. If  $I$  is a minimal left ideal of  $A$  and  $I^2 \neq 0$  then there is a projection  $p \in I$  such that  $I = Ap$ , and every such projection  $p$  is minimal [BD, Lemma 30.2]. Recall that a non-zero projection  $p$  of  $A$  is *minimal* if  $pAp$  is a division algebra.

Let  $\text{Min}(A)$  be the set of minimal projections of  $A$ . If  $A$  is semiprime then  $\text{soc}(A)$  equals the sum of all minimal right ideals of  $A$  (i.e. the *right socle*) [BD, Proposition 30.10] and  $L$  is a left (right) minimal ideal of  $A$  if and only if  $L = Ap$  (respectively,  $L = pA$ ) for some  $p \in \text{Min}(A)$ .

**Remark 6.1.** *If  $B \in \mathfrak{U}_n$  is a division algebra then  $B$  is isomorphic to  $\mathbb{C}$  by the Gelfand-Mazur theorem. If  $A$  is a semiprime Banach algebra then  $\text{soc}(A)$  is closed if and only if  $A$  is finite-dimensional [T],  $\overline{\text{soc}(A)} \cap \text{Rad}(A) = 0$  [GR, Lemma 4] and  $\text{soc}(A)$  is the largest ideal of algebraic elements of  $A$  [GR, Theorem 5].*

**Proposition 6.2.** (1) *The map  $\text{soc}$  is a preradical on  $\mathfrak{U}_a$ .*

(2) *If  $A \in \mathfrak{U}_a$  is semiprime then  $\text{soc}(J) = J \cap \text{soc}(A)$  for each ideal  $J$  of  $A$ ; in particular,  $\text{rad}(A) \cap \text{soc}(A) = 0$ .*

*Proof.* (1) is straightforward.

(2) Let  $E = \{p \in \text{Min}(A) : Jp \neq 0\}$ . Clearly  $J$  is a semiprime algebra, so the minimal left ideals of  $J$  are determined by  $\text{Min}(J)$ . Let  $p \in \text{Min}(J)$  and  $L = Jp$ . Then  $L = Jp \subset Ap \subset L$  because  $p = pp \in Jp = L$ . Hence  $Jp = Ap$  is a left ideal of  $A$  and  $p \in \text{Min}(A)$  because  $pJp = pAp$  is a division algebra. So  $Jp$  is a minimal left ideal of  $A$ . This proves that  $\text{soc}(J) \subset J \cap \text{soc}(A)$  and  $\text{Min}(J) \subset E$ . Let us prove the converse inclusions.

If  $p \in E$  then  $Ap$  is a minimal left ideal of  $A$ . As  $Jp \subset Ap$  is a non-zero left ideal of  $A$ , it follows that  $Jp = Ap$ , whence  $pJp = pAp$  is a division algebra. Therefore  $E \subset \text{Min}(J)$ . We proved that

$$\text{Min}(J) = \{p \in \text{Min}(A) : Jp \neq 0\}. \quad (6.1)$$

Let  $a \in J \cap \text{soc}(A)$  be arbitrary. Then  $a = \sum b_i p_i + \sum c_j p'_j$  (both sums are finite) for all  $b_i, c_j \in A$ , where all  $p_i \in \text{Min}(J)$  and all  $p'_j \in \text{Min}(A)$  with  $Jp'_j = 0$ . In other words,  $a = b + c$ , where

$$b := \sum b_i p_i = \sum (b_i p_i) p_i \in \text{soc}(J)$$

and  $c := \sum c_j p'_j = a - b \in J$ . By condition,  $Jp'_j = 0$  for all  $j$ , so one has  $Jc = 0$ . But the set  $K = \{d \in J : Jd = 0\}$  is an ideal of  $A$  with  $K^2 = 0$ , whence  $K = 0$  and therefore  $c = 0$  and  $a = b \in \text{soc}(J)$ . Thus  $J \cap \text{soc}(A) \subset \text{soc}(J)$ .

If  $\text{rad}(A) \cap \text{soc}(A) \neq 0$  then  $\text{rad}(A)$  has a non-zero socle and there is a non-zero projection  $p$  in  $\text{rad}(A)$ ; this is impossible.  $\square$

Let  $P$  be a preradical. Define the map  $P^{\text{soc}}$  by

$$P^{\text{soc}} = \text{soc} * P,$$

so  $P^{\text{soc}}$  is the preimage of  $\text{soc}(A/P(A))$  in  $A$ , for each algebra  $A$ . Let

$$\text{Min}_P(A) = \{x \in A : q_{P(A)}(x) \in \text{Min}(A/P(A))\}$$

where  $q_{P(A)} : A \rightarrow A/P(A)$  is the standard quotient map.

**Theorem 6.3.** *Let  $P$  be a radical such that  $P \geq \mathfrak{P}_\beta$ . Then*

- (1)  $\overline{P^{\text{soc}}}$  is a topological under radical;
- (2) If  $P$  is topological then  $\overline{P^{\text{soc}}} = P^{\overline{\text{soc}}}$ ;
- (3) If  $P$  is algebraic then  $P^{\text{soc}}$  is an algebraic under radical, and if  $P$  is hereditary then  $P^{\text{soc}*}$  is hereditary.

*Proof.* In (2) the equality  $\overline{P^{\text{soc}}} = P^{\overline{\text{soc}}}$  is evident; (1) is a consequence of (3) and Theorem 3.5 in the algebraic case and is proved similarly to (3) otherwise. So we will prove only (3).

(3)  $P^{\text{soc}}$  is a preradical by Lemma 4.10. Let  $A$  be an algebra, and let  $J$  be an ideal of  $A$ . Since  $A/P(A)$  and  $J/P(J)$  are semiprime then, to prove the inclusion  $P^{\text{soc}}(J) \subset P^{\text{soc}}(A)$ , it suffices to show that

$$\text{Min}_P(J) \subset \text{Min}_P(A) \cup \{0\}. \quad (6.2)$$

Let  $q' : J \rightarrow J/P(J)$  be the quotient map,  $q = q_{P(A)}$  (as above), and let  $g : J/P(J) \rightarrow A/P(A)$  be the homomorphism  $q'(a) \mapsto q(a)$  for  $a \in J$ . Let  $e \in \text{Min}_P(J)$  be arbitrary. Then  $q'(eJe)$  is a division algebra. If  $q(e) \neq 0$  then  $q(eJe) = g(q'(eJe))$  is also a division algebra and  $q(e) \in \text{Min}(q(J)) \subset \text{Min}(A/P(A))$  by (6.1). In all cases  $e \in \text{Min}_P(A) \cup \{0\}$ . This completes the proof of (6.2), so the inclusion  $P^{\text{soc}}(J) \subset P^{\text{soc}}(A)$  holds.

If  $a \in A$  is arbitrary, then  $q(a)q(Je) \subset q(Je)$  and  $q(Je)q(a) \subset q(Jea) = q(Jeea)$ . As  $q(ea) \in q(J)$  then, by [BD, Lemma 30.7],  $q(Je(ea))$  is a minimal left ideal of  $q(J)$  or zero. So  $P^{\text{soc}}(J)$  is an ideal of  $A$ .

Let  $B = P^{\text{soc}}(A)$ . Then  $P(B) \subset P(A)$ . But  $P(A)$  is a  $P$ -radical ideal of  $B$ ; so  $P(B) = P(A)$ . Hence  $\text{soc}(B/P(B)) = \text{soc}(A/P(A))$ . This shows that  $P^{\text{soc}}(P^{\text{soc}}(A)) = P^{\text{soc}}(A)$ . We proved that  $P^{\text{soc}}$  is an under radical.

Let now  $P$  be hereditary. Since  $P(J) = J \cap P(A)$  then  $g : q'(x) \mapsto q(x)$  is an isomorphism of  $J/P(J)$  onto the ideal  $q(J)$  of  $A/P(A)$ . Therefore

$$g(\text{Min}(J/P(J))) = \text{Min}(q(J)) = \{q(y) \in \text{Min}(A/P(A)) : q(Jy) \neq 0\}$$

by (6.1). So if  $y \in J$  and  $q(y) \in \text{Min}(A/P(A))$  then  $q'(y) \in \text{Min}(J/P(J))$ . It follows that  $J \cap P^{\text{soc}}(A) \subset P^{\text{soc}}(J)$ . Then  $P^{\text{soc}}$  is hereditary. By Corollary 4.23,  $P^{\text{soc}*}$  is hereditary.  $\square$

As a consequence of Theorem 6.3,  $P^{\text{soc}*}$  and  $\overline{P^{\text{soc}}}^*$  are (algebraic and, respectively, topological) radicals.

## 6.2. Some applications.

**Lemma 6.4.** *Let  $A$  be an algebra, and let  $I$  be a one-sided ideal of  $A$ . Then*

- (1)  $\text{rad}(I) = I \cap \text{rad}(A)$ .
- (2) *If  $J$  is a one-sided ideal of  $I$  then  $\text{rad}(J) = J \cap \text{rad}(A)$ .*

*Proof.* (1) Let  $I$  be a left ideal of  $A$ . As  $\text{rad}(A)$  is the largest left ideal consisting of left quasi-invertible elements of  $A$  [BD, Proposition 24.16],  $\text{rad}(I) \subset \text{rad}(A)$  and also  $I \cap \text{rad}(A) \subset \text{rad}(I)$ .

(2) By (1), we have that  $\text{rad}(J) = J \cap \text{rad}(I)$ . Then

$$\text{rad}(J) = J \cap \text{rad}(I) = J \cap I \cap \text{rad}(A) = J \cap \text{rad}(A).$$

$\square$

**Proposition 6.5.** *Let  $A$  be an algebra, and let  $J$  be a one-sided ideal of  $A$ . Then  $\text{rad}^{\text{soc}}(J) \subset \text{rad}^{\text{soc}}(A)$ .*

*Proof.* Let  $J$  be a left ideal. As  $\text{rad}(J) = J \cap \text{rad}(A)$  by Lemma 6.4, the standard map  $g : J/\text{rad}(J) \rightarrow B := A/\text{rad}(A)$  defined by  $g : x/\text{rad}(J) \mapsto x/\text{rad}(A)$  is one-to-one. Let  $I = g(J/\text{rad}(J))$ . Then  $I$  is a left ideal of  $B$  and one may identify the socle of  $J/\text{rad}(J)$  with the socle of  $I$  (because  $g$  and  $g^{-1}$  are morphisms).

If  $L$  is a minimal left ideal of  $I$  there is  $p \in \text{Min}(I)$  such that  $L = Ip$ . Then  $p \in L$ ,  $BL$  is a left ideal of  $B$  and  $BL \subset L$ . So either  $BL = 0$  or  $BL = L$ . But the equality  $BL = 0$  is impossible since  $B$  is semisimple. So  $L$  is a minimal left ideal of  $B$ . This proves  $\text{soc}(I) \subset \text{soc}(B)$  and the result follows.  $\square$

Recall that  $F(A)$  is the set of all finite rank elements of  $A$ .

**Lemma 6.6.** *Let  $A$  be a semiprime normed algebra. Then*

- (1)  $\overline{\text{soc}(A)} = \overline{F(A)}$ ;
- (2)  $\text{soc}(A) \cap \text{rad}(A) = 0$ ;
- (3) *If  $a, b \in F(A)$  then  $L_a R_b$  is a finite rank operator on  $A$ .*

*Proof.* (1) Let  $p \in \text{Min}(A)$  be arbitrary. As  $pAp$  is a division normed algebra,  $pAp$  is one-dimensional by the Gelfand-Masur theorem, so  $p \in F(A)$ , whence  $Ap \subset F(A)$  and  $\text{soc}(A) \subset F(A)$ .

Let  $x \in F(A)$ . Then the algebra  $xA^1x$  is finite-dimensional. As  $A^1x$  is a left ideal of  $A^1$  then  $\text{rad}(A^1x) = A^1x \cap \text{rad}(A)$  by Lemma 6.4. Let  $B = xA^1x + \mathbb{C}x$ . Then  $B$  is a finite-dimensional algebra and  $B \subset F(A)$ .

As  $B$  is a right ideal of  $A^1x$ , it follows that

$$\text{rad}(B) = B \cap \text{rad}(A) \subset F(A) \cap \text{rad}(A) = 0$$

by Lemmas 6.4 and 4.32. As  $B$  is a finite-dimensional semisimple algebra then  $B = \text{soc}(B)$ . Therefore  $x \in L_1 + \dots + L_n$  where  $L_i$  is a minimal left ideal of  $B$  for every  $i$ . Then there is  $p_i \in \text{Min}(B)$ , for every  $i$ , such that  $L_i = Bp_i$ . As

$$p_i A^1 p_i = p_i (p_i A^1 p_i) p_i \subset p_i B p_i \subset p_i A^1 p_i$$

and  $p_i B p_i = \mathbb{C} p_i$  then  $p_i \in \text{Min}(A^1)$  and  $A^1 p_i$  is a minimal left ideal of  $A^1$ , whence  $x \in \text{soc}(A^1)$ . It remains to note that if  $A$  is infinite-dimensional then clearly  $\text{soc}(A^1) = \text{soc}(A)$ , and if  $A$  is finite-dimensional then  $A$  is semisimple and  $A = A^1$  by the classical Wedderburn results.

(2) Follows from (1) and Lemma 4.32.

(3) One may assume that  $A$  is unital. Let  $p_1, p_2 \in \text{Min}(A)$  be arbitrary. As  $p_1 + p_2 \in \text{soc}(A)$  then  $L_{p_1+p_2} R_{p_1+p_2}$  is a finite rank operator, whence  $L_{p_1} R_{p_2} + L_{p_2} R_{p_1}$  is a finite rank operator on  $A$ . Then

$$L_{1-p_2} (L_{p_1} R_{p_2} + L_{p_2} R_{p_1}) = L_{p_1} R_{p_2} - L_{p_2} L_{p_1} R_{p_2}$$

is a finite rank operator. As  $L_{p_2} L_{p_1} R_{p_2} = (L_{p_2} R_{p_2}) L_{p_1}$  is a finite rank operator,  $L_{p_1} R_{p_2}$  is a finite rank operator.

Let  $a, b \in \text{soc}(A)$ . Then there are  $p_1, \dots, p_n \in \text{Min}(A)$ ,  $x_1, y_1, \dots, x_n, y_n \in A$  such that  $a = \sum_i x_i p_i$  and  $b = \sum_j y_j p_j$ . Then

$$L_a R_b = \sum_{i=1}^n \sum_{j=1}^n L_{x_i p_i} R_{y_j p_j} = \sum_{i=1}^n \sum_{j=1}^n L_{x_i} (L_{p_i} R_{p_j}) R_{y_j}$$

is a finite rank operator.  $\square$

**Remark 6.7.** (1) and (3) of Lemma 6.6 were proved for not necessary normed, semiprime algebras with lower socle instead of socle in [BE, Lemma 3.1 and Theorem 3.3]; the lower socle of  $A$  is defined as the ideal generated by all minimal projections  $p$  with  $pAp < \infty$ . It should be noted that our proofs are completely different from ones in [BE].

Lemma 6.6(3) yields

**Corollary 6.8.** *Every finite semiprime normed algebra is bifinite.*

It follows that if  $A$  is a semiprime normed algebra and  $\mathcal{R}_{\text{hf}}(A) \neq 0$  then  $\mathcal{E}(A)$  has nonzero finite rank operators. The converse is also true.

**Theorem 6.9.** *Let  $A$  be an algebra. If there is a finite rank elementary operator  $T$  on  $A$  then the image of  $T$  is contained in  $\mathfrak{R}_{\text{hf}}(A)$ .*

*Proof.* Let  $T = \sum_{i=1}^n L_{a_i} R_{b_i}$ ,  $I = \mathfrak{R}_{\text{hf}}(A)$  and  $B = A/I$ . Then the operator  $S = \sum_{i=1}^n L_{a_i/I} R_{b_i/I}$  is a finite rank elementary operator on  $B$ . By [BT, Lemma 7.1], if  $S \neq 0$  then there is a non-zero finite rank element of  $B$  which is impossible. Therefore  $S = 0$  and the image of  $T$  is contained in  $\mathfrak{R}_{\text{hf}}(A)$ .  $\square$

In particular, if  $A$  is normed then  $\mathfrak{R}_{\text{hf}}(A) \subset \mathcal{R}_{\text{hf}}(A)$  and the image of  $T$  lies in  $\mathcal{R}_{\text{hf}}(A)$ .

**Remark 6.10.** By [BT, Theorem 8.4], if  $A$  is a Banach algebra and  $T$  is a compact elementary operator on  $A$  then the image of  $T$  is contained in  $\mathcal{R}_{\text{hc}}(A)$ . It should be noted that this result holds also for normed algebras, with the same proof.

**Problem 6.11.** *Let  $A$  be a Banach algebra, and let  $T = \sum_{i=1}^{\infty} L_{a_i} R_{b_i}$  be a compact operator on  $A$  with  $\sum_i \|a_i\| \|b_i\| < \infty$  for some  $a_i, b_i \in A$ . Is the image of  $T$  contained in  $\mathcal{R}_{\text{hc}}(A)$ ?*

**Theorem 6.12.**  $\mathcal{R}_{\text{hf}} = \mathcal{P}_{\beta}^{\overline{\text{soc}}^*}$ .



*Proof.* Let  $A$  be a normed algebra. If  $A$  is  $\mathcal{P}_\beta^{\overline{\text{soc}}^*}$ -semisimple then  $A$  is semiprime and  $\text{soc}(A) = 0$ . By Lemma 6.6,  $A$  has no non-zero finite rank elements, whence  $A$  is  $\mathcal{R}_{\text{hf}}$ -semisimple. Therefore  $\mathcal{R}_{\text{hf}} \leq \mathcal{P}_\beta^{\overline{\text{soc}}^*}$  by Theorem 2.3.

Conversely, if  $A$  has no non-zero finite rank elements then  $A$  is semisimple and  $\text{soc}(A) = 0$ . Therefore  $\mathcal{P}_\beta^{\overline{\text{soc}}^*} \leq \mathcal{R}_{\text{hf}}$  by Theorem 2.3.  $\square$

**Theorem 6.13.**  $\mathcal{R}_{\text{hc}} = \mathcal{R}_{\text{hf}} \vee \mathcal{R}_{\text{jhc}} = \mathcal{R}_{\text{jhc}}^{\overline{\text{soc}}^*} \leq \mathcal{R}_{\text{cq}}^{\overline{\text{soc}}^*}$  on Banach algebras.

*Proof.* Let  $A$  be a Banach algebra. If  $A$  is  $\mathcal{R}_{\text{jhc}}^{\overline{\text{soc}}^*}$ -semisimple then  $A$  has no non-zero compact elements which lie in  $\text{Rad}(A)$ , and  $\text{soc}(A) = 0$ . Hence  $A$  is semiprime, so  $A$  has no non-zero finite rank elements by Lemma 6.6. Thus  $A$  is  $(\mathcal{R}_{\text{hf}} \vee \mathcal{R}_{\text{jhc}})$ -semisimple.

It follows that  $A$  has no non-zero compact elements. Indeed, if  $a \in A$  is a compact element then every spectral projection of  $a$  corresponding to a non-zero eigenvalue is clearly a finite rank element. So  $a$  is quasinilpotent. But  $A^1 a$  consists of compact elements, whence  $A^1 a \subset \text{Rad}(A)$ . By the assumption,  $a = 0$ . Thus  $A$  is  $\mathcal{R}_{\text{hc}}$ -semisimple.

By Theorem 2.3,

$$\mathcal{R}_{\text{jhc}}^{\overline{\text{soc}}^*} \leq \mathcal{R}_{\text{hf}} \vee \mathcal{R}_{\text{jhc}} \leq \mathcal{R}_{\text{hc}}$$

on Banach algebras. Assume now that  $A$  has no non-zero compact elements. Then it is clear that  $A$  is semiprime, has no non-zero finite rank elements and  $\text{soc}(A) = 0$ . This proves the converse.

Let now  $A$  be  $\mathcal{R}_{\text{cq}}^{\overline{\text{soc}}^*}$ -semisimple. As  $\mathcal{R}_{\text{cq}}(A) = 0$  then  $A$  is semiprime, and as  $\text{soc}(A) = 0$  then  $A$  has no non-zero finite rank elements by Lemma 6.6. As above, if  $a \in A$  is a compact element then  $a \in \text{Rad}(A)$ . But

$$\mathcal{R}_{\text{hc}}(A) \cap \text{Rad}(A) \subset \mathcal{R}_{\text{cq}}(A) = 0$$

by (2.9) (the inclusion follows by the algebraic version of the joint spectral radius formula (2.8)), whence  $a = 0$ . Therefore  $A$  has no non-zero compact elements, i.e.,  $A$  is  $\mathcal{R}_{\text{hc}}$ -semisimple. By Theorem 2.3,  $\mathcal{R}_{\text{hc}} \leq \mathcal{R}_{\text{cq}}^{\overline{\text{soc}}^*}$  on Banach algebras.  $\square$

## 7. THE KERNEL-HULL CLOSURES OF RADICALS AND THE PRIMITIVITY PROCEDURE

**7.1. The kernel-hull closures of radicals.** Let  $A$  be an algebra;  $\text{Prim}(A)$  is called the *structure space* of  $A$ . For any subset  $E$  of  $A$ , let  $\text{hull}(E; A)$  or simply  $\text{h}(E; A)$  be the set of all  $I \in \text{Prim}(A)$  with  $E \subset I$ ; this set is called a *hull* of  $E$ . For any set  $W \subset \text{Prim}(A)$ , let  $\ker(W; A)$  or  $\text{k}(W; A)$  be defined by  $\text{k}(W; A) = \bigcap_{I \in W} I$ ; this ideal is called a *kernel* of  $W$ . Note that the Jacobson radical  $\text{rad}$  coincides with the *kernel-hull closure* (briefly, *kh-closure*) of zero:

$$\text{rad}(A) := \bigcap \text{Prim}(A) = \text{kh}(\{0\}; A^1) := \text{k}(\text{h}(\{0\}; A^1); A^1)$$

for each  $A \in \mathfrak{A}_a$ . The operation of the *hull-kernel closure*

$$M \mapsto \text{hk}(M; A) := \text{h}(\text{k}(M; A); A),$$

for  $M \subset \text{Prim}(A)$ , determines the *Jacobson topology* on  $\text{Prim}(A)$  [Bo, Section 1.1]: closed sets are the sets of form  $\text{hk}(M; A^1)$ . As is known, this topology is not Hausdorff in general.

Let  $\Omega$  be a primitive map, and let  $\Pi_\Omega$  be the related ideal map on  $\mathfrak{A}$  (Section 2.3.2). Let  $A$  be an algebra, and let

$$\text{Irr}_\Omega(A) = \{\pi \in \text{Irr}(A) : \ker \pi \in \Omega(A)\}.$$

For every ideal  $J$  of  $A$ , put

$$\text{h}_\Omega(J; A) = \text{h}(J; A) \cap \Omega(A).$$

**Lemma 7.1.** *Let  $A$  be an algebra and  $I, J$  be ideals of  $A$ . Then*

- (1) *If  $I \subset J$  then  $\text{kh}_\Omega(I; A) \subset \text{kh}_\Omega(J; A)$ ;*
- (2)  *$\text{kh}_\Omega(I \cap J; J) = J \cap \text{kh}_\Omega(I; A)$ ;*
- (3)  *$\text{kh}_\Omega(J; A) = q_J^{-1}(\Pi_\Omega(A/J))$ ;*
- (4) *If  $A$  is a  $Q$ -algebra then  $I \subset \bar{I} \subset \text{kh}_\Omega(I; A) = \text{kh}_\Omega(\bar{I}; A)$ .*

*Proof.* (1) It is clear that  $\text{kh}(I; A) \subset \text{kh}(J; A)$ , whence  $\text{kh}_\Omega(I; A) \subset \text{kh}_\Omega(J; A)$ .

(2) Let  $K = I \cap J$ , and let  $\tau \in \text{Irr}_\Omega(J)$  be arbitrary with  $\tau(K) = 0$ . Then, for any representation  $\pi \in \text{Irr}_\Omega(A)$  extending  $\tau$  (on the same representation space), we have that

$$\pi(I)\tau(J) = \pi(IJ) = \tau(IJ) = 0.$$

It follows that  $\pi(I) = 0$ . Conversely, the restriction  $\tau$  of each  $\pi \in \text{Irr}_\Omega(A)$  with  $\pi(I) = 0$  to  $J$  vanishes on  $K$  and if  $\tau \neq 0$  then  $\tau \in \text{Irr}_\Omega(J)$ .

If  $a \in J \cap \text{kh}_\Omega(I; A)$  then  $\pi(a) = 0$  for every  $\pi \in \text{Irr}_\Omega(A)$  with  $\pi(I) = 0$ . For  $\tau \in \text{Irr}_\Omega(J)$  corresponding to  $\pi$ , we have that  $\tau(K) = 0$  and  $\tau(a) = \pi(a) = 0$ , whence  $a \in \text{kh}_\Omega(K; J)$ .

If  $a \in \text{kh}_\Omega(K; J)$  then  $\tau(a) = 0$  for every  $\tau \in \text{Irr}_\Omega(J)$  with  $\tau(K) = 0$ . For  $\pi \in \text{Irr}_\Omega(A)$  corresponding to  $\tau$ , we have that  $\pi(I) = 0$  and either  $\pi(J) = 0$  (whence  $\pi(a) = 0$ ) or  $\pi|_J \in \text{Irr}_\Omega(J)$  with  $\pi(a) = \tau(a) = 0$ , whence  $a \in J \cap \text{kh}_\Omega(I; A)$ .

(3) is obvious.

(4) As each  $J \in \text{h}_\Omega(I; A)$  is closed then  $\bar{I} \subset J$ , whence  $\bar{I} \subset \text{kh}_\Omega(I; A)$ .  $\square$

Let  $\Omega$  be a primitive map, and let  $P$  be a preradical. Define the map  $P^{\text{kh}_\Omega}$  by

$$P^{\text{kh}_\Omega}(A) = \text{kh}_\Omega(P(A); A)$$

for every algebra  $A$ . It follows from Lemma 7.1(3) that

$$P^{\text{kh}_\Omega}(A) = (\Pi_\Omega * P)(A) = q_{P(A)}^{-1}(\Pi_\Omega(A/P(A))) \quad (7.1)$$

for every algebra  $A$ .

In the following theorem and corollary we assume that  $\Omega$  is a primitive map defined on a base class  $\mathfrak{U}$ , and one of the following conditions, listed in Theorem 2.10, holds:

- (1<sub>kh</sub>)  $\Omega$  is pliant and either  $\mathfrak{U} = \mathfrak{U}_a$  or  $\mathfrak{U}_q \subset \mathfrak{U} \subset \mathfrak{U}_n$  (correspondingly,  $\Pi_\Omega$  is either a hereditary radical or a hereditary preradical);
- (2<sub>kh</sub>)  $\mathfrak{U} \subset \mathfrak{U}_q$  ( $\Pi_\Omega$  is a hereditary topological radical).

**Theorem 7.2.** *Let  $\Omega$  be a primitive map satisfying (1<sub>kh</sub>) or (2<sub>kh</sub>), and let  $P$  be a preradical. Then*

- (1)  *$P^{\text{kh}_\Omega}$  is a preradical, and  $P^{\text{kh}_\Omega} = \overline{P}^{\text{kh}_\Omega}$  on  $Q$ -algebras;*
- (2) *If  $P$  is an under radical then  $P^{\text{kh}_\Omega}$  is an under radical;*
- (3) *If  $P$  is hereditary then  $P^{\text{kh}_\Omega}$  is hereditary;*
- (4) *If  $\Omega$  and  $P$  are pliant then  $P^{\text{kh}_\Omega}$  is pliant.*

*Proof.* (1) If  $f : A \rightarrow B$  is a morphism then  $f(P(A)) \subset P(B)$  and  $f(P(A))$  is an ideal of  $B$ . We will show that

$$f(\text{kh}_\Omega(P(A); A)) \subset \text{kh}_\Omega(f(P(A)); B).$$

Indeed, let  $\pi \in \text{Irr}_\Omega(B)$  be arbitrary and  $\pi(P(B)) = 0$ . Then  $\tau = \pi \circ f \in \text{Irr}_\Omega(A)$  and  $\tau(P(A)) = 0$ . Then  $\tau(a) = 0$  for any  $a \in \text{kh}_\Omega(P(A); A)$ , whence  $\pi(f(a)) = 0$ . Therefore  $f(a) \in \text{kh}_\Omega(P(B); B)$ .

As  $\text{kh}_\Omega(f(P(A)); B) \subset \text{kh}_\Omega(P(B); B)$  by Lemma 7.1 then

$$f(P^{\text{kh}_\Omega}(A)) = f(\text{kh}_\Omega(P(A); A)) \subset \text{kh}_\Omega(P(B); B) = P^{\text{kh}_\Omega}(B)$$

and  $P^{\text{kh}_\Omega}$  is a preradical.

If  $A$  is a  $Q$ -algebra then

$$P^{\text{kh}\Omega}(A) = \text{kh}_\Omega(P(A); A) = \text{kh}_\Omega(\overline{P(A)}; A) = \text{kh}_\Omega(\overline{P}(A); A) = \overline{P}^{\text{kh}\Omega}(A)$$

by Lemma 7.1(4).

It remains to note that primitive ideals of a  $Q$ -algebra are closed. Hence  $P^{\text{kh}\Omega}$  is topological on  $Q$ -algebras.

(2) follows from (7.1) and Theorem 4.11.

(3) Let  $A$  be an algebra, and let  $J$  be an ideal of  $A$ . If  $P(J) = J \cap P(A)$  then

$$P^{\text{kh}\Omega}(J) = \text{kh}_\Omega(P(J); J) = J \cap \text{kh}_\Omega(P(A); A) = J \cap P^{\text{kh}\Omega}(A)$$

by Lemma 7.1, whence  $P^{\text{kh}\Omega}$  is hereditary.

(4) is obvious.  $\square$

**Corollary 7.3.** *Let  $\Omega$  be a primitive map satisfying  $(1_{kh})$  or  $(2_{kh})$ , and let  $P$  be an under radical. Then*

- (1)  $P^{\text{kh}\Omega*} = \Pi_\Omega \vee P = \Pi_\Omega^* \vee P^*$ ;
- (2)  $P^{\text{kh}\Omega*}$  is a radical, and  $P^{\text{kh}\Omega*} = \overline{P}^{\text{kh}\Omega*}$  on  $Q$ -algebras;
- (3) If  $P$  is a hereditary preradical then  $\overline{P}^{\text{kh}\Omega*}$  is hereditary on  $Q$ -algebras.
- (4) If  $\Omega$  and  $P$  are pliant then  $P^{\text{kh}\Omega*}$  is pliant.

*Proof.* (1) is clear.

(2) Assume that  $P$  is an (algebraic/topological) under radical. It is clear that

$$P^{\text{kh}\Omega*} = \text{rad} \vee P = (\text{rad} \vee P) \vee \text{rad} = (P^{\text{kh}\Omega} * \text{rad})^*.$$

Let  $(R_\alpha)$  and  $(S_\alpha)$  be the algebraic and topological convolution chains of under radicals generated by  $P^{\text{kh}\Omega} * \Pi_\Omega$  and by  $\overline{P}^{\text{kh}\Omega} * \Pi_\Omega$ , respectively.

Let  $A$  be a  $Q$ -algebra. By Theorem 7.2(1),

$$P^{\text{kh}\Omega} * \Pi_\Omega(A) = \overline{P}^{\text{kh}\Omega} * \Pi_\Omega(A).$$

We prove by transfinite induction that  $R_{\alpha+1}(A) = S_{\alpha+1}(A)$  for every ordinal  $\alpha$ , and  $R_\alpha^{\text{kh}\Omega}(A) = S_\alpha^{\text{kh}\Omega}(A)$  for a limit ordinal  $\alpha$ . Indeed, the step  $\alpha \mapsto \alpha + 1$  is easy if  $R_\alpha(A) = S_\alpha(A)$ . Assume that  $R_\alpha^{\text{kh}\Omega}(A) = S_\alpha^{\text{kh}\Omega}(A)$  for some limit ordinal  $\alpha$ . Then

$$\begin{aligned} R_{\alpha+1}(A) &= P^{\text{kh}\Omega} * \Pi_\Omega(R_\alpha(A)) = P^{\text{kh}} * \Pi_\Omega(R_\alpha^{\text{kh}}(A)) \\ &= \overline{P}^{\text{kh}} * \Pi_\Omega(S_\alpha^{\text{kh}}(A)) = \overline{P}^{\text{kh}} * \Pi_\Omega(S_\alpha(A)) \\ &= S_{\alpha+1}(A). \end{aligned}$$

So we proved the step  $\alpha \mapsto \alpha + 1$ .

So it remains to show that  $R_\alpha^{\text{kh}\Omega}(A) = S_\alpha^{\text{kh}\Omega}(A)$  for each limit ordinal  $\alpha$ . Assume by induction, that  $R_{\alpha'+1}(A) = S_{\alpha'+1}(A)$  for every ordinal  $\alpha' < \alpha$ . Then

$$\begin{aligned} R_\alpha^{\text{kh}\Omega}(A) &= \text{kh}_\Omega(\cup_{\alpha' < \alpha} R_{\alpha'+1}(A); A) = \text{kh}_\Omega(\cup_{\alpha' < \alpha} S_{\alpha'+1}(A); A) \\ &= \text{kh}_\Omega(\overline{\cup_{\alpha' < \alpha} S_{\alpha'+1}(A)}; A) = S_\alpha^{\text{kh}}(A) \end{aligned}$$

by Lemma 7.1(4).

Therefore there is an ordinal  $\gamma$  such that

$$P^{\text{kh}\Omega*}(A) = R_{\gamma+1}(A) = R_{\gamma+2}(A) = S_{\gamma+2}(A) = S_{\gamma+1}(A) = \overline{P}^{\text{kh}\Omega*}(A).$$

(3) follows from (2) and Corollary 4.23.

(4) follows from Theorem 7.2(4).  $\square$

**Remark 7.4.** In (3) of Corollary 7.3 we state the equality of two radicals on  $Q$ -algebras if one of them is obtained by application of the algebraic convolution procedure while the other one is obtained by application of the topological convolution procedure. So the resulting radical can be considered as an algebraic radical as well as a topological radical, simultaneously. This gathers all advantages of both classes of radicals.

**7.2. Primitivity procedure.** Let  $\Omega$  be a primitive map, and let  $R$  be a preradical. Define  $R^{p\Omega}$  by

$$R^{p\Omega}(A) = \cap \{(R * I; A) : I \in \Omega(A) \cup \{A\}\}$$

for every algebra  $A$ ; we recall that

$$(R * I; A) = q_I^{-1}(P(A/I)).$$

In other words,  $R^{p\Omega}(A)$  consists of all elements  $a \in A$  such that  $a/I \in P(A/I)$  for all  $I \in \Omega(A)$ .

If  $\Omega(A) = \text{Prim}(A)$  then we write  $R^p$  instead of  $R^{p\Omega}$ . The procedure  $R \mapsto R^{p\Omega}$  is called the  $\Omega$ -primitivity procedure.

**Remark 7.5.** If  $R \leq \Pi_\Omega$  then  $A/I$  is  $\Pi_\Omega$ -semisimple and therefore  $R$ -semisimple, whence  $(R * I; A) = I$  for every  $I \in \Omega(A) \cup \{A\}$ ; thus  $R^p(A) = \Pi_\Omega(A)$ . In particular this is the case for  $R \leq \text{rad}$ .

For  $R^{p\Omega}$  to have sufficiently convenient properties, one has to impose additional requirements on  $R$ . One of possible approaches is to require that  $R$  well behaves on Banach ideals. We mean the condition of Banach heredity (see (2.2)).

**Lemma 7.6.** Let  $R$  be a preradical on  $\mathfrak{U}_b$  satisfying the condition of Banach heredity. If  $A$  is a Banach algebra,  $I$  and  $J$  are closed ideals of  $A$ , then

$$(R * (J \cap I); J) = J \cap (R * I; A).$$

*Proof.* Let  $K = (J + I)/I$  with the norm  $\|\cdot\|_K$  defined in Lemma 1.7. Then the natural map  $\phi : J/(J \cap I) \rightarrow (K; \|\cdot\|_K)$  is an isometry. Let  $q : J \rightarrow J/(J \cap I)$  and  $q_I : A \rightarrow A/I$  be standard quotient maps. Considering  $\phi$  as an algebraic morphism, one may write that  $q_I|_J = \phi \circ q$ . By the assumption,

$$\phi(R((J/(J \cap I)))) = R(K; \|\cdot\|_K) = K \cap R(A/I)$$

As  $q_I|_J = \phi \circ q$  then

$$q^{-1}(R((J/(J \cap I)))) = (q_I|_J)^{-1}(K \cap R(A/I)).$$

It is clear that  $(q_I|_J)^{-1}(K) = (J + I) \cap J = J$ . Then

$$J \cap (q_I|_J)^{-1}(R(A/I)) = J \cap q_I^{-1}(R(A/I))$$

and the result follows.  $\square$

Another way to investigate the primitivity procedure is connected with modifications of Axiom 1. Namely, we may require that  $R$  be either pliant or strict (i.e. satisfies (2.1)); clearly if  $\mathfrak{U} = \mathfrak{U}_b$  then this holds for all preradicals.

In the following theorem and corollary we join both approaches.

**Theorem 7.7.** Let  $\Omega$  be a primitive map on  $\mathfrak{U}$  satisfying  $(1_{kh})$  or  $(2_{kh})$ , and let  $R$  be a preradical on  $\mathfrak{U}$ . Then

- (1) If  $\Omega$  and  $R$  are pliant then
  - (a)  $R^{p\Omega}$  is a pliant preradical;
  - (b) If  $R$  is a radical then  $R^{p\Omega}$  satisfies Axiom 4;
  - (c) If  $R$  is an hereditary preradical then  $R^{p\Omega}$  is a hereditary preradical.
- (2) If  $R$  is strict,  $\mathfrak{U}$  is universal and contained in  $\mathfrak{U}_q$  then

- (a)  $R^{p\Omega}$  is a strict preradical;
  - (b) If  $R$  is a topological radical then  $R^{p\Omega}$  satisfies Axiom 4;
  - (c) If  $R$  is a hereditary topological preradical then  $R^{p\Omega}$  is a hereditary topological preradical.
- (3) If  $R$  is defined on  $\mathfrak{U}_b$  and satisfies the condition of Banach heredity then  $R^{p\Omega}$  is a hereditary preradical on  $\mathfrak{U}_b$ .

*Proof.* (1a) & (2a) Let  $f : A \rightarrow B$  be an algebraic morphism [bounded epimorphism]. As  $(\pi \circ f) \in \text{Irr}_\Omega(A)$  for every  $\pi \in \text{Irr}_\Omega(B)$ , then  $J := f^{-1}(I) \in \Omega(A)$  for every  $I \in \Omega(B)$ . Then  $f$  induces an algebraic morphism [bounded epimorphism]  $f_0 : A/J \rightarrow B/I$  such that  $f_0 \circ q_J = q_I \circ f$ . Hence

$$\begin{aligned} f(R * J) &= f(q_J^{-1}(R(A/J))) \subset q_I^{-1}(f_0(R(A/J))) \\ &\subset q_I^{-1}(R(B/I)) = R * I \end{aligned} \quad (7.2)$$

for every  $I \in \Omega(B)$ . So  $f(R^{p\Omega}(A)) \subset R^{p\Omega}(B)$ .

Assume that  $\mathfrak{U}$  is universal and  $R$  satisfies (2.1) for continuous isomorphisms. Let  $f : A \rightarrow B$  be a continuous isomorphism. Then  $f^{-1} : B \rightarrow A$  is an algebraic morphism and  $f_0 : A/J \rightarrow B/I$  is a continuous isomorphism, where  $J = f^{-1}(I)$ . Then

$$f(R * J) = f((f_0 \circ q_J)^{-1}(f_0(R(A/J)))) = q_I^{-1}((R(A/J))) = R * I.$$

As  $\Omega(A) = \{f^{-1}(I) : I \in \Omega(B)\}$  then

$$\begin{aligned} f(R^{p\Omega}(A)) &= \cap_{I \in \Omega(B) \cup \{B\}} f(R * f^{-1}(I); A) \\ &= \cap_{I \in \Omega(B) \cup \{B\}} (R * I; B) = R^{p\Omega}(B). \end{aligned}$$

Therefore  $R^{p\Omega}$  satisfies (2.1) for continuous isomorphisms.

(1b) & (2b) If  $K$  is an ideal of  $A$  then  $\Omega(K) \cup \{K\} = \{K \cap I : I \in \Omega(A) \cup \{A\}\}$ . As  $K \cap I$  is an ideal of  $A$  then  $(R * (K \cap I); K)$  is an ideal of  $A$ . Therefore  $R^{p\Omega}(K)$  is an ideal of  $A$ .

Let  $q'_I : K \rightarrow K/(K \cap I)$  and  $q_I : A \rightarrow A/I$  be standard quotient maps. There is a [bounded] injective imbedding  $g_I$  of  $K/(K \cap I)$  onto an ideal  $q_I(K)$  of  $A/I$ , whence

$$g_I(R(K/(K \cap I))) \subset R(q_I(K)) \subset R(A/I). \quad (7.3)$$

As  $q_I|_K = g_I \circ q'_I$  then

$$\begin{aligned} R^{p\Omega}(K) &= \cap_{I \in \Omega(A) \cup \{A\}} (q'_I)^{-1}(R(K/(K \cap I))) \\ &= \cap_{I \in \Omega(A) \cup \{A\}} (g_I \circ q'_I)^{-1}(g_I(R(K/(K \cap I)))) \\ &\subset \cap_{I \in \Omega(A) \cup \{A\}} g_I^{-1}(R(A/I)) = R^{p\Omega}(A). \end{aligned} \quad (7.4)$$

Therefore  $R^{p\Omega}$  satisfies Axiom 4.

(1c) & (2c) As  $R$  is hereditary, and also is algebraic or satisfies (2.1) for continuous isomorphisms, then we rewrite (7.3) as follows:

$$g_I(R(K/(K \cap I))) = R(q_I(K)) = q_I(K) \cap R(A/I)$$

for every  $I \in \Omega(A) \cup \{A\}$ . It follows from (7.4) that

$$\begin{aligned} R^{p\Omega}(K) &= \cap_{I \in \Omega(A) \cup \{A\}} (q_I|_K)^{-1}(q_I(K) \cap R(A/I)) \\ &= \cap_{I \in \Omega(A) \cup \{A\}} (K \cap q_I^{-1}(R(A/I))) \\ &= K \cap (\cap_{I \in \Omega(A) \cup \{A\}} q_I^{-1}(R(A/I))) = K \cap R^{p\Omega}(A). \end{aligned}$$

Therefore  $R^{p\Omega}$  is a hereditary preradical.

(3) Let  $f : A \rightarrow B$  be a topological morphism of Banach algebras. Then  $(\pi \circ f) \in \text{Irr}_\Omega(A)$  for every  $\pi \in \text{Irr}_\Omega(B)$ , whence  $J := f^{-1}(I) \in \Omega(A) \cup \{A\}$  for

every  $I \in \Omega(B) \cup \{B\}$ . Then  $f$  induces a topological morphism  $f_0 : A/J \rightarrow B/I$  such that  $f_0 \circ q_J = q_I \circ f$ . Hence

$$f(R * J) \subset R * I$$

as in (7.2), for every  $I \in \Omega(B) \cup \{B\}$ . So  $f(R^{p\Omega}(A)) \subset R^{p\Omega}(B)$ .

By our assumption, for a closed ideal  $K$  of  $A$ ,

$$\begin{aligned} R^{p\Omega}(K) &= \bigcap_{I \in \Omega(A) \cup \{A\}} (R * (K \cap I); K) \\ &= \bigcap_{I \in \Omega(A) \cup \{A\}} K \cap (R * I; A) \\ &= K \cap \left( \bigcap_{I \in \Omega(A) \cup \{A\}} (R * I; A) \right) = K \cap R^{p\Omega}(A) \end{aligned}$$

by Lemma 7.6.  $\square$

Now we apply the convolution procedure to obtain a radical.

**Corollary 7.8.** *Let  $\Omega$  be a primitive map on  $\mathfrak{A}$  satisfying  $(1_{kh})$  or  $(2_{kh})$ , and let  $R$  be a preradical. Then*

- (1) *If  $\Omega$  and  $R$  are pliant then  $R^{p\Omega*}$  is a pliant hereditary radical;*
- (2) *If  $R$  is a closed ideal map satisfying (2.1) (i.e.  $R$  is strict),  $\mathfrak{A}$  is universal and contained in  $\mathfrak{A}_q$  then  $R^{p\Omega*}$  is a topological radical on  $\mathfrak{A}$ ;*
- (3) *If  $R$  is defined on  $\mathfrak{A}_b$  and satisfies the condition of Banach heredity then  $R^{p\Omega*}$  is a radical on Banach algebras;*
- (4)  $\Pi_\Omega \vee R = \Pi_\Omega^* \vee R^* \leq R^{p\Omega*}$ .

*Proof.* (1)-(3) follow from the properties of convolution procedure.

(4) It is clear that

$$\Pi_\Omega(A) = \bigcap_{I \in \Omega(A) \cup \{A\}} q_I^{-1}(0) \subset \bigcap_{I \in \Omega(A) \cup \{A\}} q_I^{-1}(R(A/I)) = R^{p\Omega}(A)$$

for an algebra  $A$ , and every  $R$ -radical algebra is  $R^{p\Omega}$ -radical. This implies the result.  $\square$

## 8. SCATTERED ALGEBRAS

Our aim in this section is to give a consistent exposition of the theory of scattered radical. This radical was announced in [ST2] and already applied in [BT, ST6].

**8.1. Thin ideals.** Let  $K \subset \mathbb{C}$  be a bounded set. Recall that the *polynomially convex hull*  $\text{pc}(K)$  of  $K$  is the set of all  $z \in \mathbb{C}$  such that

$$|p(z)| \leq \sup_{w \in K} |p(w)|$$

for every polynomial  $p$ , and  $K$  is *polynomially convex* if  $K = \text{pc}(K)$ . By [G, Lemma 3.1.3], a compact subset  $K \subset \mathbb{C}$  is polynomially convex if and only if the complement of  $K$  is connected. We say that  $H$  is a *polynomially convex neighborhood* of  $K$  if  $K$  is contained in the interior of  $H$  and  $H = \text{pc}(H)$  (hence  $H$  is a compact subset of  $\mathbb{C}$ ).

An algebra  $A$  is called *inessential* (or *Riesz* [BMS]) if  $\sigma(a)$  is finite or countable with the only one limit point at 0 for every  $a \in A$ ;  $A$  is *scattered* if  $\sigma(a)$  is at most countable for every  $a \in A$ . These notions are transferred to ideals. As ideals are spectral subalgebras, it is not important with respect to what – the algebra or the ideal – the spectrum of an element of the ideal is considered. The algebras  $\mathcal{F}(X)$  of finite rank operators and  $\mathcal{K}(X)$  of compact operators on a Banach space  $X$  are inessential, and they are inessential ideals of  $\mathcal{B}(X)$ .

By [A, Corollary 5.7.6], if  $I$  is an inessential ideal of a Banach algebra  $A$  then  $\overline{I}$  and  $\text{kh}(I; A)$  are also inessential ideals of  $A$  and have the same set of projections as  $I$ .

**Lemma 8.1.** *Let  $A$  be a Banach algebra, and let  $I$  be an inessential ideal of  $A$ . If  $a \in A$  and  $V$  is a polynomially convex neighborhood of  $\sigma(a/I)$  then there is a finite set  $Z$  such that  $\sigma(a) \subset V \cup Z$ .*

*Proof.* Let  $\Delta$  be the set of all isolated points in  $\sigma(a)$  with Riesz (spectral) projections in  $I$ , and  $D = \sigma(a) \setminus \Delta$ . By [A, Theorem 5.7.4] and Corollary 1.5,

$$D \subset \text{pc}(\sigma(a/\bar{I})) = \text{pc}(\sigma(a/I)).$$

Then  $D \subset V$ , and only a finite number of points in  $\sigma(a)$  can be outside of  $V$  because otherwise  $\sigma(a)$  would have a limit point outside of  $D$  which contradicts the definition of  $D$ . Therefore

$$\sigma(a) \subset V \cup Z$$

for some finite set  $Z \subset \mathbb{C}$ . □

Let us say that an ideal  $I$  of an algebra  $A$  is *thin* if for each  $a \in A$  there is a countable set  $Z \subset \mathbb{C}$  such that

$$\sigma(a) \subset \text{pc}(\sigma(a/I)) \cup Z. \quad (8.1)$$

**Lemma 8.2.** *Let  $A$  be a [normed] algebra, and let  $J \subset I$  be [closed] ideals of  $A$ . If  $I$  is a thin ideal of  $A$  then  $J$  is a thin ideal of  $A$ .*

*Proof.* It is sufficient to note that  $\sigma(a/I) \subset \sigma(a/J)$  for every  $a \in A$ . □

If  $I$  is a thin ideal of a  $Q$ -algebra then  $\bar{I}$  is also thin by Corollary 1.5. Recall that the spectrum of each element of a  $Q$ -algebra is a compact set in  $\mathbb{C}$ .

**Lemma 8.3.** *Let  $A$  be a  $Q$ -algebra. An ideal  $I$  of  $A$  is thin if and only if for each  $a \in A$  and for every polynomially convex neighborhood  $V$  of  $\sigma(a/I)$ , there is a countable set  $Z \subset \mathbb{C}$  such that  $\sigma(a) \subset V \cup Z$ .*

*Proof.*  $\Rightarrow$  is obvious.

$\Leftarrow$  Let  $a \in A$ , and let  $Z_V := \sigma(a) \setminus V$  be countable for each polynomially convex neighborhood  $V$  of  $\sigma(a/I)$ . Let  $D \subset \mathbb{C}$  be an open disk with

$$\bar{D} \cap \text{pc}(\sigma(a/I)) = \emptyset.$$

As  $\text{pc}(\sigma(a/I))$  is polynomially convex, its complement  $C$  is connected, so it contains  $E_t = \{z : |z| \geq t\}$  for sufficiently large  $t > 0$ , and a curve  $L \subset C$  connecting  $D$  with  $E_t$ . Let  $U \subset C$  be a connected neighborhood of  $E_t \cup L \cup \bar{D}$ . Then the complement  $V$  of  $U$  is a compact neighborhood of  $\sigma(a/I)$ . Since the complement of  $V$  is connected then  $V$  is polynomially convex. By the assumption,

$$\sigma(a) \subset V \cup Z_V$$

where  $Z_V$  is countable (or finite) by assumption.

The set  $\mathbb{C} \setminus \text{pc}(\sigma(a/I))$  is open and has therefore an inscribed covering by open disks. Taking an inscribed subcovering by disks with rational radii and coordinates of centers one can find a sequence  $(D_k)_{k=1}^\infty$  of open disks such that

$$\bigcup_{k=1}^\infty D_k = \mathbb{C} \setminus \text{pc}(\sigma(a/I)) \text{ and all } \bar{D}_k \cap \text{pc}(\sigma(a/I)) = \emptyset;$$

let  $V_k$  be compact neighborhoods of  $\text{pc}(\sigma(a/I))$  non-intersecting  $D_k$ . Then, as each  $V_k$  lies in the complement  $D_k^c$  of  $D_k$ ,

$$\text{pc}(\sigma(a/I)) \subset \bigcap_{k=1}^\infty V_k \subset \bigcap_{k=1}^\infty D_k^c \subset (\bigcup_{k=1}^\infty D_k)^c = \text{pc}(\sigma(a/I)),$$

and

$$\sigma(a) \subset \bigcap_{k=1}^\infty (V_k \cup Z_{V_k}) \subset (\bigcap_{k=1}^\infty V_k) \cup (\bigcup_{k=1}^\infty Z_{V_k}) = \text{pc}(\sigma(a/I)) \cup Z$$

where  $Z = \bigcup_{k=1}^\infty Z_{V_k}$  is a countable set. □

Taking into account Lemma 8.1, we see that inessential ideals of Banach algebras are thin ideals.

**Corollary 8.4.** *Let  $A$  be a  $Q$ -algebra. Then each thin ideal  $J$  is a scattered algebra.*

*Proof.* Indeed, if  $a \in J$  then  $\sigma(a/J) = \{0\}$ , whence, by (8.1),  $\sigma(a) \subset \{0\} \cup Z$  is countable.  $\square$

**Lemma 8.5.** *Let  $A$  be a  $Q$ -algebra, and let  $I \subset J$  be closed ideals of  $A$ . If  $I$  and  $J/I$  are thin ideals of  $A$  and  $A/I$ , respectively, then  $J$  is a thin ideal of  $A$ .*

*Proof.* Let  $a \in A$  be arbitrary. Since the standard morphism  $\tau : A/I \rightarrow A/J$  has the kernel  $J/I$  then, by Lemma 8.3,

$$\sigma(a/I) \subset \text{pc}(\sigma(a/J)) \cup Z$$

where  $Z$  is a countable set. As the union of a polynomially convex set and a (bounded) countable set in  $\mathbb{C}$  is polynomially convex then

$$\text{pc}(\sigma(a/I)) \subset \text{pc}(\sigma(a/J)) \cup Z.$$

Since  $I$  is thin then

$$\sigma(a) \subset \text{pc}(\sigma(a/I)) \cup Z'$$

where  $Z'$  is countable. Thus

$$\sigma(a) \subset \text{pc}(\sigma(a/J)) \cup Z \cup Z'.$$

$\square$

The following quite general result establishes a kind of spectral continuity.

**Lemma 8.6.** *Let  $A$  be a  $Q$ -algebra, let  $(J_\alpha)_{\alpha \in \Lambda}$  be an up-directed net of closed ideals of  $A$  and  $J = \overline{\bigcup_{\alpha \in \Lambda} J_\alpha}$ . Then*

(1)

$$\sigma(a/J) = \bigcap_{\alpha \in \Lambda} \sigma(a/J_\alpha); \quad (8.2)$$

(2) *If all ideals  $J_\alpha$  are thin then  $J$  is thin.*

*Proof.* (1) If  $\lambda \notin \sigma(a/J)$  then there are  $b \in A$  and  $x \in J$  with  $(a - \lambda)b = 1 + x$ . Hence one can find  $\alpha \in \Lambda$  and  $y \in J_\alpha$  with  $\|x - y\| < 1$ . By [P, Proposition 2.2.7],  $\sum_{n>0} (y - x)^n$  converges in  $A$ . Therefore  $1 - (y - x)$  is invertible and  $1 + (x - y) = z^{-1}$  for some  $z \in A$ , whence  $(a - \lambda)b = z^{-1} + y$  and

$$(a - \lambda)bz = 1 + yz.$$

Therefore  $\lambda \notin \sigma^l(a/J_\alpha)$ . Similarly, there is  $\delta \in \Lambda$  with  $\lambda \notin \sigma^r(a/J_\delta)$ . So  $\lambda \notin \sigma(a/J_\gamma)$  for  $\gamma > \alpha$  and  $\gamma > \delta$ . Hence

$$\bigcap_{\alpha \in \Lambda} \sigma(a/J_\alpha) \subset \sigma(a/J).$$

The converse inclusion is evident.

(2) Let  $V$  be a polynomially convex neighborhood of  $\sigma(a/J)$  then, by (8.2) and compactness of  $\sigma(a/J)$ , there is  $\alpha \in \Lambda$  with  $\sigma(a/J_\alpha) \subset V$ . Since  $J_\alpha$  is thin there is a countable set  $Z \subset \mathbb{C}$  such that  $\sigma(a) \subset V \cup Z$ . It remains to apply Lemma 8.3.  $\square$

**Theorem 8.7.** *Let  $A$  be a  $Q$ -algebra, and let  $(J_\alpha)_{\alpha \leq \gamma}$  be an increasing transfinite chain of closed ideals of  $A$  with  $J_0 = 0$ . If each ideal  $J_{\alpha+1}/J_\alpha$  is a thin ideal of  $A/J_\alpha$  then all ideals  $J_\alpha$  are thin ideals of  $A$  and therefore scattered algebras.*



*Proof.* Using transfinite induction, assume that for increasing transfinite chains of ideals with final ordinals  $< \alpha$  the statement is true, i.e. all  $J_{\alpha'}$  with  $\alpha' < \alpha$  are thin ideals. If  $\alpha$  is a limit ordinal then the result follows from Lemma 8.6(2). Otherwise  $\alpha = \alpha' + 1$  for some  $\alpha'$  and the result follows from Lemma 8.5 applied to ideals  $J_{\alpha'} \subset J_{\alpha'+1}$ . Therefore all ideals  $J_\alpha$  are thin. By Corollary 8.4, they are also scattered.  $\square$

**8.2. Scattered radical.** Recall that the algebraic under radical  $\text{rad}^{\text{soc}} = \text{soc} * \text{rad}$  sends each algebra  $A$  to the ideal  $\{x \in A : x / \text{rad}(A) \in \text{soc}(A / \text{rad}(A))\}$ . This ideal, the socle modulo radical, was called in [BMS, Definition F.3.1] by the *presocle* of  $A$  and denoted by  $\text{psoc}(A)$ . We preserve this notation for the map itself:  $\text{psoc} := \text{soc} * \text{rad}$ .

Now we can apply the algebraic convolution procedure and obtain the hereditary algebraic radical  $\text{psoc}^*$  on  $\mathfrak{U}_a$ .

**Theorem 8.8.** *The restriction of  $\text{psoc}^*$  to  $\mathfrak{U}_q$  is a hereditary topological radical on  $Q$ -algebras.*

*Proof.* As  $\text{rad} \leq \text{psoc}$  then

$$\text{rad} \vee \text{psoc} = \text{psoc}^*.$$

(Recall that  $\text{rad} \vee \text{psoc}$  is the smallest radical that is larger than or equal to  $\text{rad}$  and  $\text{psoc}$ .) One can realize  $\text{rad} \vee \text{psoc}$  as the action of the algebraic convolution procedure, namely

$$\text{rad} \vee \text{psoc} = (\text{rad} * \text{psoc})^*.$$

But  $\text{rad} * \text{psoc} = \text{psoc}^{\text{kh}}$ . So

$$\text{psoc}^* = \text{psoc}^{\text{kh}*}.$$

By Corollary 7.3,  $\text{psoc}^{\text{kh}*}$  is a hereditary topological radical on  $Q$ -algebras.  $\square$

Define map  $\mathcal{R}_s$  by

$$\mathcal{R}_s = \text{psoc}^* \text{ on Banach algebras;}$$

this map is a hereditary topological radical on  $\mathfrak{U}_b$ ; it is called the *scattered radical*.

**Lemma 8.9.** *Let  $A$  be a Banach algebra. Then  $\mathcal{R}_s(A)$  is a thin ideal of  $A$  and a scattered algebra.*

*Proof.* Let  $(R_\alpha)_\alpha$  be the convolution chain of under radicals generated by  $\overline{\text{psoc}}$ . As  $\overline{\text{psoc}}^* = \mathcal{R}_s$  by Corollary 7.3, then there is  $\gamma$  such that  $R_\gamma(A) = \mathcal{R}_s(A)$  and all gap-quotients  $R_{\alpha+1}(A) / R_\alpha(A) = \overline{\text{psoc}}(A / R_\alpha(A))$  are inessential ideals by [BMS, Section R], and therefore thin ideals by Lemmas 8.1 and 8.3. By Theorem 8.7,  $\mathcal{R}_s(A)$  is a thin ideal of  $A$  and a scattered algebra.  $\square$

Let  $\mathcal{S}(A)$  be the set of all elements of  $A$  with (finite or) countable spectrum.

**Theorem 8.10.** *Let  $A$  be a Banach algebra. Then*

- (1)  $\mathcal{R}_s(A)$  contains all one-sided, non-necessarily closed, scattered ideals of  $A$ ;
- (2)  $\mathcal{R}_s(A)$  is the largest thin ideal and the largest scattered ideal of  $A$ ;
- (3)  $\mathcal{R}_s(A) = \{a \in A : \sigma(ax) \text{ is countable for each } x \in A\}$ ;
- (4)  $\mathcal{R}_s(A)$  is the largest ideal of  $A$  contained in  $\mathcal{S}(A)$ ;
- (5)  $\mathcal{R}_s(A) + \mathcal{S}(A) \subset \mathcal{S}(A)$ .

*Proof.* (1) Let  $J$  be a scattered left ideal in  $A$ . Let  $B = A / \mathcal{R}_s(A)$ , and let  $I$  be the image of  $J$  in  $B$ . Then  $B$  is semisimple and  $I$  is a scattered left ideal of  $B$ . If it is non-zero then it contains an element  $a$  with non-zero spectrum and therefore there is an isolated non-zero point  $\lambda$  in its spectrum. As  $\sigma_B(a) \subset \sigma_I(a)$  and  $\lambda \in \sigma_I^a(a) \subset \sigma_B^a(a)$ ,  $\lambda$  is an isolated point of  $\sigma_B(a)$ . As  $\lambda \neq 0$ , the corresponding Riesz projection  $p$  of  $a$  belongs to  $Ba \subset I$  (see for instance the proof of Lemma 5.7.1

of  $[A]$ ). As  $p \neq 0$  then  $Bp$  is a closed non-zero scattered left ideal of  $B$ . By Barnes' Theorem [B],  $\text{psoc}(Bp) \neq 0$ . By Proposition 6.5,  $\text{psoc}(B) \neq 0$ . Thus  $\mathcal{R}_s(B) \neq 0$ , a contradiction. Therefore  $I = 0$ , whence  $J \subset \mathcal{R}_s(A)$ .

(2) It follows from (1) that  $\mathcal{R}_s(A)$  is the *largest scattered ideal* of  $A$ . Let  $I$  be a thin ideal of  $A$ . Then it is a scattered ideal of  $A$  by Corollary 8.4, and so  $I \subset \mathcal{R}_s(A)$ .

(3) Let  $a \in A$  and  $J = Aa$ . If  $a \in \mathcal{R}_s(A)$  then  $J$  is contained in  $\mathcal{R}_s(A)$  and its elements have countable spectra. Conversely, if  $J$  is scattered then it is contained in  $\mathcal{R}_s(A)$  by Proposition 8.10.

(4) If  $I$  is an ideal of  $A$  contained in  $\mathcal{S}(A)$  then it is a scattered ideal of  $A$ . By (2),  $I \subset \mathcal{R}_s(A)$ .

(5) Let  $a \in \mathcal{S}(A)$  and  $b \in \mathcal{R}_s(A)$ . As  $\mathcal{R}_s(A)$  is a thin ideal of  $A$  then

$$\sigma(a + b) \subset \text{pc}(\sigma(a/\mathcal{R}_s(A))) \cup N$$

for some countable set  $N \subset \mathbb{C}$ . But  $\sigma(a/\mathcal{R}_s(A))$  is countable, whence

$$\text{pc}(\sigma(a/\mathcal{R}_s(A))) = \sigma(a/\mathcal{R}_s(A))$$

is countable and  $a + b \in \mathcal{S}(A)$ . □

**Corollary 8.11.**  $\mathcal{R}_s$  is a uniform radical on  $\mathfrak{U}_b$ .

*Proof.* Indeed, every closed subalgebra of a scattered Banach algebra is scattered. □

**Corollary 8.12.** The radical  $\mathcal{R}_s$  satisfy the condition of Banach heredity (2.2).

*Proof.* Let  $L$  be a Banach ideal of a Banach algebra  $A$ . This means that  $L$  is an ideal of  $A$  and there is an injective continuous homomorphism  $f$  of a Banach algebra  $B$  to  $A$  with  $f(B) = L$ . Since  $f$  is an isomorphism of algebras  $B$  and  $L$ , and  $\text{psoc}^*$  is a hereditary radical on  $\mathfrak{U}_a$ , we have that

$$\begin{aligned} \mathcal{R}_s((L, \|\cdot\|_B)) &= f(\mathcal{R}_s(B)) = f(\text{psoc}^*(B)) = \text{psoc}^*(L) \\ &= L \cap \text{psoc}^*(A) = L \cap \mathcal{R}_s(A). \end{aligned}$$

□

**Corollary 8.13.** Let  $A$  be a Banach algebra, and let  $I$  be a (non-necessary closed) ideal of  $A$ . Then

- (1) If  $I$  is scattered then  $I$  is a thin ideal of  $A$ ;
- (2) If  $I$  and  $A/I$  are scattered then  $A$  is scattered;
- (3) If  $f : A \rightarrow B$  is an algebraic morphism of Banach algebras then

$$f(\mathcal{R}_s(A)) \subset \mathcal{R}_s(B).$$

*Proof.* (1) Indeed,  $\bar{I} \subset \mathcal{R}_s(A)$  by Theorem 8.10. As  $\mathcal{R}_s$  is hereditary,  $\bar{I} = \mathcal{R}_s(\bar{I})$ . Taking  $(R_\alpha)$  as in Lemma 8.9 we see that all  $R_\alpha(I)$  are thin ideals of  $A$ . Therefore  $\mathcal{R}_s(\bar{I})$  is a thin ideal of  $A$ , hence  $\bar{I}$  and  $I$  are also thin.

(2) It follows that  $\bar{I}$  and  $A/\bar{I}$  are also scattered. Then  $\bar{I} = \mathcal{R}_s(\bar{I})$  and  $A/\bar{I} = \mathcal{R}_s(A/\bar{I})$ . By Theorem 4.6,  $A = \mathcal{R}_s(A)$ . Therefore  $A$  is scattered.

(3) Indeed,  $\mathcal{R}_s$  inherits the algebraic properties of  $\text{psoc}^*$ . □

**Theorem 8.14.** If an algebra  $A$  is a subideal of a Banach algebra (see Section 2.2) or is algebraically isomorphic to the quotient of a subideal of a Banach algebra by a non-necessarily closed ideal then

- (1)  $\text{psoc}^*(A) = I_A$  where  $I_A := \{a \in A : \sigma_A(ax) \text{ is countable for each } x \in A\}$ ;
- (2) Every scattered ideal of  $A$  is a thin ideal of  $A$ .

*Proof.* Note that every subideal of a Banach algebra is a normed  $Q$ -algebra.

Let  $C$  be a  $Q$ -algebra for which (1) and (2) hold (under the substitution  $A = C$ ; for instance these conditions hold for Banach algebras), and let  $A$  be an ideal of  $C$ . Then  $\text{psoc}^*(A) = A \cap I_C$  by heredity of  $\text{psoc}^*$ . As clearly  $I_A = A \cap I_C$  then  $\text{psoc}^*(A) = I_A$ .

Let  $K$  be a scattered ideal of  $A$ , and let  $J$  be the ideal of  $C$  generated by  $K$ . Then

$$J^3 \subset K \subset J$$

by [AR, Lemma 1.1.5], whence  $J^3$  is a scattered ideal of  $C$ . Then  $J^3$  is a thin ideal of  $C$  by (2) and therefore a thin ideal of  $A$ . As  $K^3 \subset J^3$  then  $K^3$  and  $\overline{K^3}$  are thin ideals of  $A$  by Lemma 8.2. As  $\overline{K}/\overline{K^3}$  is nilpotent then it is a thin ideal of  $A/\overline{K^3}$ . By Lemma 8.5,  $\overline{K}$  is a thin ideal of  $A$ , so  $K$  is also a thin ideal of  $A$ .

Thus (1) and (2) hold for  $A$ : therefore the steps  $0 \mapsto 1$  and  $n \mapsto n+1$  of induction for  $n$ -subideals of Banach algebras are valid, for every  $n$ . Then the proof is completed for subideals of Banach algebras.

Let now  $f : A \rightarrow B/J$  be an algebraic isomorphism of  $A$  onto  $B/J$ , where  $B$  is a subideal of a Banach algebra and  $J$  is an ideal of  $B$ . If  $a \in A$  and  $b \in B$  is any element such that  $f(a) = b/J$ , then clearly  $\sigma(a) = \sigma(b/J)$  and it suffices to check (1) and (2) for  $B/J$ . As  $\sigma(b/J) = \sigma(b/\overline{J})$  for every  $b \in B$  by Lemma 1.5, the proof is reduced to the case of  $B/\overline{J}$ . But  $B/\overline{J}$  is a subideal of a Banach algebra (see for instance [ST5, Theorem 2.24]), and the result follows from the above.  $\square$

Theorem 8.14 extends the main properties of the scattered radical from Banach algebras to subideals of Banach algebras. Recall that subideals of Banach algebras form the smallest universal class  $\mathfrak{U}_b^u$  generated by Banach algebras. Thus we extend the denotation  $\mathcal{R}_s$  for  $\text{psoc}^*$  on subideals of Banach algebras.

The *regular scattered radical*  $\mathcal{R}_s^r$ , obtained from  $\mathcal{R}_s$  by the regular procedure, extends  $\mathcal{R}_s$  to normed algebras, and it is determined as

$$\mathcal{R}_s^r(A) = \left\{ x \in A : \sigma_{\widehat{A}}(ax) \text{ is at most countable } \forall a \in \widehat{A} \right\}.$$

**Theorem 8.15.**  $\mathcal{R}_{\text{hc}} \leq \mathcal{R}_s = \mathcal{R}_{\text{hf}} \vee \text{Rad} = \mathcal{R}_{\text{hc}} \vee \text{Rad}$  on Banach algebras.

*Proof.* Let  $A$  be a Banach algebra. Any compact element of a Banach algebra  $A$  has countable spectrum [Al] and any bicomact Banach algebra is scattered. Then any closed bicomact ideal of  $A$  is scattered and contained in  $\mathcal{R}_s(A)$ , whence  $\Sigma_{\text{hc}}(A) \subset \mathcal{R}_s(A)$  and  $\Sigma_{\text{hc}} \leq \mathcal{R}_s$  in general. As  $\mathcal{R}_s$  is a radical, it follows that  $\mathcal{R}_{\text{hf}} = \mathcal{R}_{\text{hc}} = \Sigma_{\text{hc}}^* \leq \mathcal{R}_s$  by Theorems 3.1 and 3.9.

It is clear that  $\text{Rad} \leq \mathcal{R}_s$ , so  $\mathcal{R}_{\text{hf}} \vee \text{Rad} \leq \mathcal{R}_s$ .

Let  $A$  be  $(\mathcal{R}_{\text{hf}} \vee \text{Rad})$ -semisimple, and let  $I = \mathcal{R}_s(A)$ . As  $A$  is semisimple and has no non-zero finite rank elements,  $I$  is also semisimple and has no non-zero finite rank elements. However, if  $I \neq 0$  then  $I$  has a non-zero socle by Barnes' theorem [B]. Thus, by Lemma 6.6,  $I$  has non-zero finite rank elements, and so  $A$  has such elements, a contradiction. Therefore  $I = 0$ , and  $A$  is  $\mathcal{R}_s$ -semisimple. By Theorem 2.3,  $\mathcal{R}_s \leq \mathcal{R}_{\text{hf}} \vee \text{Rad}$ .

Furthermore,

$$\mathcal{R}_s = \mathcal{R}_{\text{hc}} \vee \mathcal{R}_s = \mathcal{R}_{\text{hc}} \vee \mathcal{R}_{\text{hf}} \vee \text{Rad} = \mathcal{R}_{\text{hc}} \vee \text{Rad}.$$

$\square$

Let  $A$  be an algebra, and let  $\widehat{\text{Irr}}(A)$  be the set of classes of equivalent strictly irreducible representations of  $A$ . Then  $\pi \mapsto \ker \pi$  is a map from  $\widehat{\text{Irr}}(A)$  onto  $\text{Prim}(A)$ , so  $\widehat{\text{Irr}}(A)$  inherits the Jacobson topology from  $\text{Prim}(A)$ : the preimages

of closed sets in  $\text{Prim}(A)$  determine closed sets in  $\widehat{\text{Irr}}(A)$ . The following statement says that one can identify these topological spaces for scattered Banach algebras.

**Theorem 8.16.** *Let  $A$  be a Banach algebra. If  $A$  is scattered then for any primitive ideal  $I$ , there is only one, up to the equivalence, strictly irreducible representation with kernel  $I$ .*

*Proof.* Any representation with kernel  $I$  defines a faithful representation of  $A/I$ . Since  $A/I$  is a primitive Banach algebra, we have only to prove that a faithful representation of a primitive scattered algebra  $B$  is unique (up to the equivalence).

Since  $B$  is scattered and semisimple, the socle of  $B$  is non-zero. So it contains a minimal projection  $p$ . Taking into account that  $\dim pBp = 1$  and applying Lemma 1.1, we have that there is only one strictly irreducible representation  $\pi$  with  $\pi(p) \neq 0$ . But the last condition holds for each faithful representation.  $\square$

A topological space is called *dispersed* if it does not contain perfect subspaces, i.e., closed subsets without isolated points.

**Theorem 8.17.** *If a Banach algebra  $A$  is scattered then the space  $\text{Prim}(A)$  of its primitive ideals is dispersed.*

*Proof.* Let  $E$  be a closed subset in  $\text{Prim}(A)$ ,  $J = \bigcap_{I \in E} I$  and  $B = A/J$ . All primitive ideals of  $B$  are of the form  $I/J$ ,  $I \in E$  (because  $E$  is closed) and their intersection is trivial. Thus  $B$  is a semisimple scattered algebra, whence it contains a minimal projection  $p$ . Since  $\dim pBp = 1$ , there is, by Lemma 1.1, only one primitive ideal  $I'_0 = I_0/J$  of  $B$  that does not contain  $p$ . It follows that  $I_0$  does not contain the intersection of all  $I \neq I_0$  in  $E$ . Hence  $I_0$  is an isolated point in  $E$  and  $E$  is not perfect.  $\square$

**Corollary 8.18.** *If  $A$  is a separable scattered Banach algebra then the spaces  $\text{Prim}(A)$  and  $\widehat{\text{Irr}}(A)$  are countable.*

*Proof.* It follows from Theorem 8.16 that if  $\text{Prim}(A)$  is countable then  $\widehat{\text{Irr}}(A)$  is countable.

In any topological space  $X$  there is a decreasing transfinite chain of sets: let  $X_0 = X$ ,  $X_{\alpha+1}$  be the set of all non-isolated points in  $X_\alpha$  for every ordinal  $\alpha$ , and let  $X_\alpha = \bigcap_{\alpha' < \alpha} X_{\alpha'}$  if  $\alpha$  is a limit ordinal. If  $X$  is dispersed then  $X_\delta = \emptyset$  for some  $\delta$ .

Let  $X = \text{Prim}(A)$ . As  $X$  is dispersed by Theorem 8.17, let  $(X_\alpha)_{\alpha \leq \delta}$  be the chain of the above sets of  $X$ , with  $X_\delta = \emptyset$ . Choose  $I_\alpha \in X_\alpha \setminus X_{\alpha+1}$  for every ordinal  $\alpha < \delta$ . By the construction,  $I_\alpha \notin \overline{\{I_{\alpha'} : \alpha' > \alpha\}}$  in the Jacobson topology. Hence there are  $x_\alpha \in (\bigcap_{\alpha' > \alpha} I_{\alpha'}) \setminus I_\alpha$ . Multiplying by a constant, we can have

$$\text{dist}(x_\alpha, I_\alpha) > 1.$$

Let now  $\alpha' > \alpha$ . Then  $x_\alpha \in I_{\alpha'}$  and

$$\|x_{\alpha'} - x_\alpha\| \geq \text{dist}(x_{\alpha'}, I_{\alpha'}) > 1.$$

So the last inequality holds for all ordinals  $\alpha, \alpha'$  with  $\alpha \neq \alpha'$ . Since  $A$  is separable, we obtain that  $\delta$  is a countable ordinal.

Now it suffices to show that each set  $X_\alpha \setminus X_{\alpha+1}$  is countable. This can be done by the same trick because each ideal in  $X_\alpha \setminus X_{\alpha+1}$  is not contained in the closure of the set of the others.  $\square$

As an example let us look at the algebras  $C(X)$  where  $X$  is a compact set.

**Corollary 8.19.** *An algebra  $C(X)$  is scattered if and only if  $X$  is dispersed.*

*Proof.* It follows from Theorem 8.17 that if  $C(X)$  is scattered then  $X$  is dispersed. Conversely, let  $X$  be dispersed. Any quotient of  $C(X)$  by a closed ideal is isomorphic to  $C(Y)$ , where  $Y$  is a compact subset of  $X$ . Since  $Y$  is not perfect, there is an isolated point  $z \in Y$ . The ideal of all functions in  $C(Y)$  that vanish outside  $z$  is one-dimensional, hence it is thin. This allows us to construct a transfinite chain  $(I_\alpha)_{\alpha \leq \gamma}$  of closed ideals of  $C(X)$ , such that  $I_0 = 0$ ,  $I_\gamma = C(X)$  and all gap-quotients of the chain are thin ideals. By Theorem 8.7,  $C(X)$  is scattered.  $\square$

**Remark 8.20.** *The above corollary gives a proof of the fact that the image of a dispersed space under a continuous map is dispersed. Indeed, if  $Y = f(X)$  then  $C(Y)$  is isomorphic to a closed subalgebra  $B$  of  $C(X)$ . If  $X$  is dispersed then  $C(X)$  is scattered. Then  $B$  is scattered, so  $C(Y)$  is scattered. Hence  $Y$  is dispersed.*

**8.3. Scattered radical on hereditarily semisimple Banach algebras.** By Theorem 8.15, all hypocompact Banach algebras are scattered. The converse is not true in general because all Jacobson radical algebras are scattered. We will show here that if the radicals of an algebra and all its quotients are trivial that these conditions are equivalent.

Let  $A$  be a [normed] algebra. Let us call  $A$  *hereditarily semisimple* if all its quotients by [closed] ideals are semisimple. It is obvious that the class of all hereditarily semisimple [normed] algebras has the following properties:

- (1) All quotients by [closed] ideals are in this class;
- (2) If a [closed] ideal  $I$  of a [normed] algebra  $A$  and the quotient  $A/I$  are in this class then so is  $A$ .

**Proposition 8.21.** *A Banach algebra  $A$  is hereditarily semisimple if and only if each closed ideal of  $A$  is the intersection of a family of primitive ideals.*

*Proof.* Let  $J$  be a closed ideal of  $A$  and let  $K = \text{kh}(J)$ . Then  $K/J = \text{Rad}(A/J)$ . Indeed, if  $I \in \text{Prim}(A/J)$  then there is  $I' \in \text{Prim}(A)$  with  $J \subset I'$  and  $I = I'/J$ . Since  $K \subset I'$ , we have that  $K/J \subset I$ . Thus

$$K/J \subset \bigcap_{I \in \text{Prim}(A/J)} I = \text{Rad}(A/J).$$

Conversely, if  $K' = q_J^{-1}(\text{Rad}(A/J))$  then  $K' \subset I'$  for any primitive ideal  $I' \supset J$ . Hence  $K' \subset K$  and  $\text{Rad}(A/J) \subset q_J(K) = K/J$ .

Now if  $A$  is hereditarily semisimple then  $K/J = 0$ , whence  $J = K$ , the intersection of primitive ideals. Conversely, if  $\text{kh}(J) = J$  for all  $J$ , then  $\text{Rad}(A/J) = 0$ , whence  $A$  is hereditarily semisimple.  $\square$

In commutative case the condition that each closed ideal of a Banach algebra  $A$  is the intersection of primitive ideals, is sometimes formulated as  $A$  possesses the spectral synthesis. Among group algebras  $L^1(G)$ , only the algebras of compact groups have this property (Malliavin's Theorem, see [Rd]).

**Theorem 8.22.** *Let  $A$  be a hereditarily semisimple Banach algebra. Then the following conditions are equivalent:*

- (1)  $A$  is scattered;
- (2)  $A$  is hypocompact;
- (3)  $A$  is a closed-hypofinite algebra;
- (4) Every non-zero quotient of  $A$  has a minimal left ideal.

*Proof.* (1)  $\implies$  (4) Let  $A$  be scattered, and let  $J$  be a closed ideal of  $A$ . Then  $A/J$  is also scattered. So it suffices to show that a scattered semisimple Banach algebra  $A$  has non-zero socle. But if  $\text{soc}(A) = 0$  then  $\text{psoc}(A) = 0$ , whence  $\text{psoc}^*(A) = 0$ , i.e.,  $\mathcal{R}_s(A) = 0$  while  $\mathcal{R}_s(A) = A$ , a contradiction.

(4)  $\implies$  (3) follows from the fact that each minimal projection is a finite rank element.

(3)  $\implies$  (2) is evident.

(2)  $\implies$  (1) follows from Theorem 8.15.  $\square$

An example of a hereditarily semisimple algebra is the algebra  $C(X)$  of all continuous functions on a compact space  $X$ . We saw that it is scattered if and only if  $X$  does not contain perfect subsets. Hence it is hypocompact under the same condition.

Since all  $C^*$ -algebras are semisimple and their quotients are again  $C^*$ -algebras, we have that  $C^*$ -algebras are hereditarily semisimple.

Note that in the theory of  $C^*$ -algebras the term “scattered” is used for a class of  $C^*$ -algebras  $A$  satisfying the following equivalent conditions:

- (1<sub>s</sub>) Each positive functional on  $A$  is the sum of a sequence of pure functionals;
- (2<sub>s</sub>)  $A$  is a type I  $C^*$ -algebra (= GCR  $C^*$ -algebra) and  $\text{Prim}(A)$  is dispersed in the hull-kernel topology;
- (3<sub>s</sub>)  $A$  is of type I and the maximal ideal space of its center is dispersed;
- (4<sub>s</sub>)  $A$  admits a superposition series  $(I_\gamma)$  of closed ideals and each gap-quotient  $I_{\gamma+1}/I_\gamma$  is isomorphic to  $\mathcal{K}(H_\gamma)$  for some Hilbert space  $H_\gamma$ ;
- (5<sub>s</sub>) Each self-adjoint element of  $A$  has countable spectrum;
- (6<sub>s</sub>) Each  $C^*$ -subalgebra of  $A$  is AF (approximately finite-dimensional);
- (7<sub>s</sub>) The dual space  $A^*$  of  $A$  has the Radon-Nikodym property.

For the proof of the equivalence see [Ks] and the references therein. We will show that this class of algebras are exactly the scattered  $C^*$ -algebras.

**Theorem 8.23.** *A  $C^*$ -algebra  $A$  satisfies the equivalent conditions (1<sub>s</sub>)-(7<sub>s</sub>) if and only if it is scattered.*

*Proof.* Clearly each scattered  $C^*$ -algebra satisfies (6<sub>s</sub>) and (5<sub>s</sub>). On the other hand, if  $A$  satisfies (4<sub>s</sub>) then  $A$  is hypocompact, so it is scattered by Theorem 8.15.  $\square$

Thus taking into account that  $C^*$ -algebras are hereditary semisimple and using Theorem 8.22 one can add to the above list of the equivalent properties of a  $C^*$ -algebra the following ones:

- (8<sub>s</sub>) All elements of  $A$  have countable spectra;
- (9<sub>s</sub>) Each non-zero quotient of  $A$  has a non-zero compact element;
- (10<sub>s</sub>) Each non-zero quotient of  $A$  has a non-zero finite element;
- (11<sub>s</sub>) Every non-zero quotient of  $A$  has a minimal projection.

It was shown in [AW] that an element  $a$  of a  $C^*$ -algebra  $A$  is compact if and only if the map  $x \mapsto axa$  is weakly compact. So one can add also the condition:

- (12<sub>s</sub>) Each non-zero quotient of  $A$  has a non-zero weakly compact element.

## 9. SPECTRAL CONTINUITY AND RADICALS

Spectral continuity, that is the continuity of such functions of an operator (or an element of Banach algebra) as spectrum, special parts of spectrum, spectral radius and so on, is a very convenient property when it holds. For the information on this subject see for example [Bu] and the references therein. We consider it from the viewpoint of theory of topological radicals. An important role here is played by the scattered radical  $\mathcal{R}_s$ , its primitivity extension  $\mathcal{R}_s^p$  and the radical  $\mathcal{R}_s^{p*}$  obtained from  $\mathcal{R}_s^p$  by the convolution procedure. So we begin with a study of these ideal maps.

### 9.1. $\mathcal{R}_s^a$ , $\mathcal{R}_s^p$ , $\mathcal{R}_s^{p*}$ and continuity of the spectrum.

**Proposition 9.1.**  $\mathcal{R}_s^a$  is a hereditary topological radical.

*Proof.* Since the radical  $\mathcal{R}_s$  satisfies the condition of Banach heredity (Corollary 8.12), its centralization  $\mathcal{R}_s^a$  is a hereditary topological radical on  $\mathfrak{U}_b$ , by Theorem 5.7 and Remark 5.8.  $\square$

The Banach heredity implies, by Theorem 7.7(3), that the primitivity extension  $\mathcal{R}_s^p$  of  $\mathcal{R}_s$  is a hereditary preradical on  $\mathfrak{U}_b$ . Recall that a Banach algebra  $A$  is  $\mathcal{R}_s^p$ -radical if  $A/I$  is  $\mathcal{R}_s$ -radical (= scattered), for each primitive ideal  $I$  of  $A$ .

**Theorem 9.2.** The classes  $\mathbf{Rad}(\mathcal{R}_s^p) \setminus \mathbf{Rad}(\mathcal{R}_s^a)$  and  $\mathbf{Rad}(\mathcal{R}_s^a) \setminus \mathbf{Rad}(\mathcal{R}_s^p)$  are non-empty.

*Proof.* Let  $A = C([0, 1], \mathcal{K}(H))$  be the  $C^*$ -algebra of all norm-continuous  $\mathcal{K}(H)$ -valued functions on  $[0, 1]$ . It is well known that  $\text{Prim}(A)$  is isomorphic to  $[0, 1]$  via the map  $t \mapsto I_t$  where  $I_t$  is the ideal of all functions equal 0 at  $t$ . Since  $A/I_t \cong \mathcal{K}(H)$  then  $A$  is  $\mathcal{R}_s^p$ -radical. On the other hand,  $A$  has no ideals  $J$  with commutative  $A/J$ , so it is not  $\mathcal{R}_s^a$ -radical.

To show the non-voidness of  $\mathbf{Rad}(\mathcal{R}_s^a) \setminus \mathbf{Rad}(\mathcal{R}_s^p)$ , let us consider the Toeplitz algebra  $A_t$  that is the  $C^*$ -algebra generated by the unilateral shift  $V$  in the space  $H = l_2(\mathbb{N})$ , acting by the rule  $Ve_n = e_{n+1}$ , where  $(e_n)_{n=1}^\infty$  is the standard basis in  $l_2(\mathbb{N})$ . It is known [Dv, Theorem 5.1.5] that  $A_t$  contains the ideal  $\mathcal{K}(H)$  and that  $A_t/\mathcal{K}(H)$  is isomorphic to the algebra  $C(\mathbb{T})$  of all continuous functions on the unit circle. Since  $\mathcal{K}(H) \subset \mathcal{R}_s(A_t)$ , we get that  $A_t \in \mathbf{Rad}(\mathcal{R}_s^a)$ . On the other hand, the identity representation of  $A_t$  is irreducible, while  $A_t$  is not scattered, because  $\sigma(V) = \mathbb{D}$ , the unit disk. Therefore  $A_t$  is not  $\mathcal{R}_s^p$ -radical.  $\square$

Now let us consider the radical  $\mathcal{R}_s^{p*}$  which is obtained from  $\mathcal{R}_s^p$  by the convolution procedure. We will see later that this radical plays a very important role in continuity of the spectral radius.

**Lemma 9.3.**  $\mathcal{R}_s^a < \mathcal{R}_s^{p*}$  on  $\mathfrak{U}_b$ .

*Proof.* All commutative Banach algebras belong to  $\mathcal{R}_s^p$  because their strictly irreducible representations are one-dimensional. The quotient  $\mathcal{R}_s^a(A)/\mathcal{R}_s(A)$  is a commutative algebra, for every Banach algebra  $A$ . Thus we have that  $\mathcal{R}_s^a(A)$  is an extension of an  $\mathcal{R}_s^{p*}$ -radical algebra by an  $\mathcal{R}_s^p$ -radical algebra. Since  $\mathcal{R}_s^{p*}$  is a radical,  $\mathcal{R}_s^a(A)$  is  $\mathcal{R}_s^{p*}$ -radical. By Theorems 2.3 and 9.2,  $\mathcal{R}_s^a < \mathcal{R}_s^{p*}$ .  $\square$

Since  $A_t$  is in  $\mathbf{Rad}(\mathcal{R}_s^a) \setminus \mathbf{Rad}(\mathcal{R}_s^p)$ , we conclude that  $\mathcal{R}_s^p \neq \mathcal{R}_s^{p*}$ . In other words, we obtain the following:

**Corollary 9.4.** The under radical  $\mathcal{R}_s^p$  is not a radical.

Recall some facts about continuity of the spectrum.

Let  $A$  be a Banach algebra. It is well known that  $\sigma$  is upper continuous, i.e. if  $a_n \rightarrow a$  then for every open neighborhood  $V$  of  $\sigma(a)$  there is  $m > 0$  such that  $\sigma(a_n) \subset V$  for every  $n > m$ . Moreover, Newburgh's theorem [Sn, Lemma 6.15] says that for every clopen subset  $\sigma_0$  of  $\sigma(a)$  and every open neighborhood  $V_0$  of  $\sigma_0$ , there is  $m_0$  such that  $\sigma_n \subset V_0$  for  $n > m_0$  and some clopen subsets  $\sigma_n$  of  $\sigma(a_n)$ .

In particular, if  $\sigma(a)$  is at most countable then the spectrum  $\sigma$  is continuous at  $a$ , i.e.  $\sigma(a_n) \rightarrow \sigma(a)$  by the Hausdorff metric. For continuity of  $\sigma$  at  $a$  it is sufficient to have that for every  $\lambda \in \sigma(a)$  there is a sequence  $\lambda_n \in \sigma(a_n)$  such that  $\lambda_n \rightarrow \lambda$  as  $n \rightarrow \infty$ .

We need the following result of Zemanek [Z2, Remark 1].

**Lemma 9.5.** *Let  $A$  be a Banach algebra, and let  $a, a_n \in A$ ,  $a_n \rightarrow a$  as  $n \rightarrow \infty$ . If  $\sigma_{A/I}(a_n/I) \rightarrow \sigma_{A/I}(a/I)$ , for every  $I \in \text{Prim}(A)$ , then  $\sigma_A(a_n) \rightarrow \sigma_A(a)$ .*

It is convenient to use a more general result of this type.

**Lemma 9.6.** *Let  $\{f_\alpha : \alpha \in \Lambda\}$  be a set of homomorphisms of a Banach algebra  $A$  to some  $Q$ -algebras  $A_\alpha$ . Let  $a \in A$  be such that*

$$\sigma(a) = \overline{\bigcup_{\alpha \in \Lambda} \sigma(f_\alpha(a))}. \quad (9.1)$$

*If  $a_n \rightarrow a$  and  $\sigma(f_\alpha(a_n)) \rightarrow \sigma(f_\alpha(a))$ , for each  $\alpha \in \Lambda$ , then  $\sigma(a_n) \rightarrow \sigma(a)$ .*

*Proof.* Let  $G = \liminf_{n \rightarrow \infty} \sigma(a_n) := \{\lambda \in \mathbb{C} : \lambda = \lim_{n \rightarrow \infty} \lambda_n \text{ for some } \lambda_n \in \sigma(a_n)\}$ . By the upper continuity, it suffices to show that  $\sigma(a) \subset G$ . But  $G$  is closed and contains all  $\sigma(f_\alpha(a))$ . So the result follows from (9.1).  $\square$

Note that apart from the case  $f_\alpha = q_{I_\alpha}$ ,  $I_\alpha \in \text{Prim}(A)$ , the condition (9.1) holds in many other situations, for example when  $\Lambda$  is finite and  $\bigcap_{\alpha \in \Lambda} I_\alpha \subset \text{Rad}(A)$  (see Theorem 10.1).

**Theorem 9.7.** *Let  $A$  be a Banach algebra. Then each element  $a \in \mathcal{R}_s^p(A)$  is a point of the spectrum continuity.*

*Proof.* Let  $a \in \mathcal{R}_s^p(A)$ ,  $a_n \in A$ ,  $a_n \rightarrow a$ ; by Lemma 9.5, it suffices to show that  $\sigma(a_n/I) \rightarrow \sigma(a/I)$ , for each  $I \in \text{Prim}(A)$ . Since  $\sigma(a/I)$  is countable of finite, the fact follows from the Newburgh theorem.  $\square$

We show now that Theorem 9.7 does not transfer to  $\mathcal{R}_s^a$  and therefore to  $\mathcal{R}_s^{p*}$ .

**Theorem 9.8.** *The spectrum in an  $\mathcal{R}_s^a$ -radical Banach algebra can be discontinuous.*

*Proof.* We use the example of spectral discontinuity proposed by G. Lumer (see [H, Problem 86]). Let  $H = l^2(\mathbb{Z})$ , and let  $\{e_n : n \in \mathbb{Z}\}$  be the natural orthonormal basis in  $H$ . Let us denote by  $W$  the shift operator  $W$  on  $H$  ( $We_n = e_{n+1}$ ). For each  $k \in \mathbb{N}$ , let  $W_k$  be the operator acting on the basis by the formulas  $W_k e_n = e_{n+1}$ , for  $n \neq 0$ ,  $W_k e_0 = \frac{1}{k} e_1$ . Then  $W_k \rightarrow W_\infty$  where  $W_\infty e_n = e_{n+1}$  for  $n \neq 0$ ,  $W_\infty e_0 = 0$ . It is easy to check that  $\sigma(W_k) = \mathbb{T}$  while  $\sigma(W_\infty) = \mathbb{D}$ , the unit disk.

Since  $W_k - W$  is a rank one operator, for each  $k$ , all operators  $W_k$ ,  $k = 1, 2, \dots, \infty$ , belong to the  $C^*$ -algebra  $A$  generated by  $W$  and the ideal  $\mathcal{K}(H)$  of all compact operators. Then  $A/\mathcal{K}(H)$  is commutative (it is generated by the image of the normal operator  $W$  in the Calkin algebra  $\mathcal{B}(H)/\mathcal{K}(H)$ ). Therefore  $A \in \mathbf{Rad}(\mathcal{R}_s^a)$  since it is commutative modulo the scattered ideal  $\mathcal{K}(H)$ .  $\square$

**Corollary 9.9.** *The spectrum continuity is not a radical/semisimple property, even in Banach algebras.*

*Proof.* Indeed, assume, to the contrary, that there is a radical  $P$  on Banach algebras such that  $\mathbf{Rad}(P)$  or  $\mathbf{Sem}(P)$  coincides with the class of algebras with the property of the spectrum continuity. Then this class contains commutative algebras and scattered algebras. As  $\mathbf{Rad}(P)$  or  $\mathbf{Sem}(P)$  is stable under extensions, then it contains the algebra constructed in Theorem 9.8, a contradiction.  $\square$

In the same time,  $\mathbf{Rad}(\text{rad}^{\dim=\infty}|_{\mathfrak{A}_q})$  is contained in the class of algebras with the property of the spectrum continuity.

**Problem 9.10.** *Is there the largest radical  $P$  such that  $\mathbf{Rad}(P)$  is contained in the class of Banach algebras with the property of the spectrum continuity?*



At the moment, the largest radical  $P$  about which we know that  $P$ -radical algebras enjoy the property of spectral continuity, is  $\text{rad}^{\dim=\infty}$  (which is clearly majorized by  $\mathcal{R}_s^p$ ).

We finish by a discussion of the restrictions of the ideal maps  $\mathcal{R}_s^p$ ,  $\mathcal{R}_s^a$  and  $\mathcal{R}_s^{p*}$  to the class  $\mathfrak{U}_{c*}$ .

It was already mentioned that an irreducible  $*$ -representation of a  $C^*$ -algebra is strictly irreducible and that for each primitive ideal  $I$  of a  $C^*$ -algebra  $A$ , the quotient  $A/I$  is isometrically isomorphic to the image of  $A$  in an irreducible  $*$ -representation. These facts will be multiply used below. Moreover, dealing with  $C^*$ -algebra we usually write representation meaning  $*$ -representation.

**Theorem 9.11.** *Let  $A$  be a  $C^*$ -algebra. Then  $\mathcal{R}_s^{p*}(A) = \mathcal{R}_{\text{gcr}}(A)$ .*

*Proof.* Since  $\mathcal{R}_s^{p*}$  and  $\mathcal{R}_{\text{gcr}}$  are radicals, it suffices to show (taking into account Corollary 2.4) that  $\mathbf{Rad}(\mathcal{R}_{\text{gcr}}) = \mathbf{Rad}(\mathcal{R}_s^{p*}) \cap \mathfrak{U}_{c*}$ .

Let  $A$  be a CCR-algebra; then its image in each irreducible representation  $\pi$  coincides with  $\mathcal{K}(H_\pi)$ , for some Hilbert space  $H_\pi$ ; so  $\pi(A)$  is scattered and therefore  $\mathcal{R}_s$ -radical. Hence  $A$  is  $\mathcal{R}_s^{p*}$ -radical. Since each GCR-algebra  $B$  admits an increasing transfinite chain  $(I_\alpha)_{\alpha \leq \gamma}$  with CCR gap-quotients and  $I_0 = 0$ ,  $I_\gamma = B$ , then  $\mathbf{Rad}(\mathcal{R}_s^{p*})$  contains all GCR  $C^*$ -algebras.

Conversely, each  $\mathcal{R}_s$ -radical  $C^*$ -algebra is a GCR-algebra by Theorem 8.23. If a  $C^*$ -algebra  $A$  is  $\mathcal{R}_s^{p*}$ -radical, then, by definition, for each irreducible representation  $\pi$ ,  $\pi(A)$  is  $\mathcal{R}_s$ -radical, so it is a GCR-algebra. It follows that  $\pi(A)$  contains a non-zero compact operator. Thus  $A$  is a GCR-algebra.

Since each  $\mathcal{R}_s^{p*}$ -radical  $C^*$ -algebra  $C$  admits an increasing transfinite chain  $(J_\alpha)_{\alpha \leq \delta}$  with  $\mathcal{R}_s$ -radical gap-quotients and  $J_0 = 0$ ,  $J_\delta = C$ , then  $\mathbf{Rad}(\mathcal{R}_s^{p*}) \cap \mathfrak{U}_{c*}$  consists of GCR  $C^*$ -algebras.  $\square$

It follows from Theorem 9.11 that  $\mathcal{R}_s^{p*}$  can be considered as a natural extension of the GCR-radical from the class  $\mathfrak{U}_{c*}$  to all Banach algebras.

We saw in the proof of Theorem 9.11 that the class of all  $\mathcal{R}_s^{p*}$ -radical  $C^*$ -algebras contains the class of all CCR-algebras. The inclusion is strict: for example it is not difficult to construct a  $C^*$ -algebra  $A \subset \mathcal{B}(H)$  that contains  $\mathcal{K}(H)$  and such that  $A/\mathcal{K}(H) \cong \mathcal{K}(H)$  — it is scattered but not CCR.

The examples in Theorem 9.2 show that CCR-algebras need not be  $\mathcal{R}_s^a$ -radical, and that  $\mathcal{R}_s^a$ -radical  $C^*$ -algebras are not necessarily CCR. The full description of  $\mathcal{R}_s^a$ -radical  $C^*$ -algebras is very non-trivial: the famous work [BDF] shows that even the classification of commutative extensions of the algebra  $\mathcal{K}(H)$  is related to deep homological constructions.

Concluding this subsection let us show that spectrum is continuous *at normal elements* in a wide class of  $C^*$ -algebras that are far from being GCR — for example, in  $C^*$ -algebras of free groups.

A  $C^*$ -algebra  $A$  is called *residually finite-dimensional* (RFD, for short) if there is a family  $\{\pi_\alpha : \alpha \in \Lambda\}$  of finite-dimensional representations of  $A$  with  $\bigcap_{\alpha \in \Lambda} \ker \pi_\alpha = 0$ . Such algebras are  $\text{rad}^{\dim<\infty}$ -semisimple.

**Theorem 9.12.** *Let  $A$  be an RFD  $C^*$ -algebra. Then every normal element  $a$  of  $A$  is a point of continuity of the spectrum  $\sigma_A$ .*

*Proof.* Let  $(\pi_\alpha)_{\alpha \in \Lambda}$  be a family of finite-dimensional representations of  $A$  with  $\bigcap_{\alpha \in \Lambda} \ker \pi_\alpha = 0$ . Let  $(a_n)$  be a sequence of elements of  $A$  tending to  $a$ . Since  $\sigma(\pi_\alpha(a_n)) \rightarrow \sigma(\pi_\alpha(a))$  for each  $\alpha$ , it is sufficient to show that

$$\sigma(a) \subset \overline{\bigcup_{\alpha \in \Lambda} \sigma(\pi_\alpha(a))}$$

and apply Proposition 9.6.

Let  $\lambda_0 \in \sigma(a) \setminus \overline{\cup_{\alpha \in \Lambda} \sigma(\pi_\alpha(a))}$ , then there is a disk  $D_\varepsilon(\lambda_0) = \{\lambda : |\lambda - \lambda_0| < \varepsilon\}$  which does not intersect  $\cup_{\alpha \in \Lambda} \sigma(\pi_\alpha(a))$ . Therefore for each  $\alpha$  there is an element  $b_\alpha \in \pi_\alpha(A)$  which is inverse to  $\pi_\alpha(a - \lambda_0)$  and

$$\|b_\alpha\| = \left\| (\pi_\alpha(a - \lambda_0))^{-1} \right\| = \rho(\pi_\alpha(a - \lambda_0))^{-1} < \varepsilon^{-1}$$

since  $a$  is normal.

Let  $H = \oplus_{\alpha \in \Lambda} H_\alpha$  and  $\pi = \oplus_{\alpha \in \Lambda} \pi_\alpha$ , the direct sum of representations  $\pi_\alpha$ . Then  $\ker(\pi) = 0$  and  $B = \pi(A)$  is a  $C^*$ -algebra isomorphic to  $A$ . The element  $b = \oplus_{\alpha \in \Lambda} b_\alpha$  belongs to  $\mathcal{B}(H)$ , because  $\sup_{\alpha \in \Lambda} \|b_\alpha\| < \infty$ . It follows easily from the definition of  $b_\alpha$  that

$$b\pi(a - \lambda_0) = \pi(a - \lambda_0)b = 1.$$

Therefore  $\lambda_0 \notin \sigma_{\mathcal{B}(H)}(\pi(a))$ ; since  $\pi(A)$  is a  $C^*$ -algebra then it is an inverse-closed subalgebra of  $\mathcal{B}(H)$  and

$$\lambda_0 \notin \sigma_{\pi(A)}(\pi(a)) = \sigma(a),$$

a contradiction.  $\square$

**9.2. Continuity of the spectral radius.** It turns out that the situation with the continuity of the spectral radii is different.

Let  $A$  be a Banach algebra. Recall that the function  $\rho : a \mapsto \rho(a)$  is upper continuous because it is the infimum of continuous functions  $a \mapsto \|a^n\|^{1/n}$ .

Let  $\mathcal{V}_\rho$  be the class of all Banach algebras  $A$  satisfying the following condition

- (1 $_\rho$ ) For every Banach algebra  $B$  containing  $A$  as a closed ideal, any element  $b \in B$  with  $\rho(b) > \rho(b/A)$  is a point of continuity of the function  $\rho$  in  $B$ .

**Theorem 9.13.** *Let  $B$  be a Banach algebra, and let  $A \in \mathcal{V}_\rho$  be its closed ideal. Then  $A$  consists of points of continuity of  $\rho$  in  $B$ .*

*Proof.* Let  $a \in A$ . If  $\rho(a) > 0$  then  $a$  is a point of continuity of  $\rho$  in  $B$  by (1 $_\rho$ ). If  $\rho(a) = 0$  then continuity of  $\rho$  at  $a$  follows from the upper continuity.  $\square$

**Lemma 9.14.** *Let  $A$  be a normed algebra, and let  $(J_\alpha)_{\alpha \in \Lambda}$  be an up-directed net of closed ideals of  $A$ ,  $J = \overline{\cup_{\alpha \in \Lambda} J_\alpha}$ . Then*

$$\begin{aligned} \|M/J\| &= \lim \|M/J_\alpha\| = \inf \|M/J_\alpha\|, \\ \rho(M/J) &= \lim \rho(M/J_\alpha) = \inf \rho(M/J_\alpha) \end{aligned}$$

for every precompact set  $M$  in  $A$ .

*Proof.* This is [ST7, Lemma 4.10].  $\square$

Let  $\mathcal{F}_{\mathcal{V}_\rho}$  be the family of all under radicals whose radical classes are contained in  $\mathcal{V}_\rho$ .

**Theorem 9.15.**  *$\vee \mathcal{F}_{\mathcal{V}_\rho}$  is a radical and  $\vee \mathcal{F}_{\mathcal{V}_\rho} \in \mathcal{F}_{\mathcal{V}_\rho}$ .*

*Proof.* Indeed,  $H_{\mathcal{F}_{\mathcal{V}_\rho}}$  is an under radical by Theorem 4.1. As  $\vee \mathcal{F}_{\mathcal{V}_\rho} = H_{\mathcal{F}_{\mathcal{V}_\rho}}^*$  then  $\vee \mathcal{F}_{\mathcal{V}_\rho}$  is a radical by Theorem 3.1.

Let  $A$  be a  $(\vee \mathcal{F}_{\mathcal{V}_\rho})$ -radical Banach algebra. By Theorem 4.19, there is an increasing transfinite chain  $(I_\alpha)_{\alpha \leq \gamma}$  of closed ideals of  $A$  such that  $I_0 = 0$ ,  $I_\gamma = A$  and for each  $\alpha < \gamma$  there is  $P \in \mathcal{F}_{\mathcal{V}_\rho}$  with  $I_{\alpha+1}/I_\alpha = P(A/I_\alpha)$ .

Let  $B$  be a Banach algebra for which  $A$  is an ideal. It follows that  $I_1 = P_1(A)$  for some  $P_1 \in \mathcal{F}_{\mathcal{V}_\rho}$ . So  $I_1$  is an ideal of  $B$ . Similarly,  $I_2/I_1 = P_2(A/I_1)$  for some  $P_2 \in \mathcal{F}_{\mathcal{V}_\rho}$  and therefore  $I_2/I_1$  is an ideal of  $B/I_1$ . As  $I_2 = q_{I_1}^{-1}(P_2(A/I_1))$  then  $I_2$  is an ideal of  $B$ . Applying the transfinite induction, it is easy to check that all  $I_\alpha$  are ideals of  $B$ .

Assume, to the contrary, that  $A \notin \mathcal{V}_\rho$ . Then there are a Banach algebra  $B$  and an element  $b \in B$  with  $\rho(b) > \rho(b/A)$  which is a point of discontinuity of  $\rho$ . So there is a sequence  $(b_n) \subset B$  such that  $b_n \rightarrow b$  as  $n \rightarrow \infty$ , but  $\limsup \rho(b_n) < \rho(b)$ .

Take the first ordinal  $\alpha'$  for which  $\rho(b) \neq \rho(b/I_{\alpha'})$ . By Lemma 9.14,  $\alpha'$  is not a limit ordinal. So there is an ordinal  $\alpha < \gamma$  such that

$$\rho(b) = \rho(b/I_\alpha) > \rho(b/I_{\alpha+1}). \quad (9.2)$$

Let  $C = B/I_\alpha$ ,  $J = I_{\alpha+1}/I_\alpha$  and  $x = b/I_\alpha$ . Then one can rewrite (9.2) as

$$\rho(x) > \rho(x/J).$$

As  $J \in \mathcal{V}_\rho$  is a closed ideal of  $C$  then  $x$  is a point of continuity of  $\rho$ . Then

$$\liminf \rho(b_n) \geq \liminf \rho(b_n/I_\alpha) = \rho(b/I_\alpha) = \rho(b) > \limsup \rho(b_n),$$

a contradiction.  $\square$

The radical  $\vee \mathcal{F}_{\mathcal{V}_\rho}$  is denoted by  $\mathcal{R}_{\vec{\rho}}$  and is called the  $\rho$ -continuity radical. Now we extend our knowledge about  $\mathbf{Rad}(\mathcal{R}_{\vec{\rho}})$ .

**Theorem 9.16.**  $\mathcal{R}_s^{p*} \leq \mathcal{R}_{\vec{\rho}}$ .

*Proof.* Let  $B$  be a Banach algebra, let  $A$  be a closed  $\mathcal{R}_s^p$ -radical ideal of  $B$ , and let  $b \in B$  be such that  $\rho(b) > \rho(b/A)$ . Let a sequence  $b_n$  tend to  $b$  as  $n \rightarrow \infty$ .

Assume, to the contrary, that  $\limsup \rho(b_n) < \rho(b)$ . Choose  $\varepsilon > 0$  such that

$$\rho(b) - \varepsilon > \max \{ \rho(b/A), \limsup \rho(b_n) \}. \quad (9.3)$$

As  $\rho(b) = \sup_{I \in \text{Prim}(B^1)} \rho(b/I)$  then there is a primitive ideal  $I$  of  $B$  such that

$$\rho(b/I) > \rho(b) - \varepsilon.$$

Thus

$$\rho(b/I) > \rho(b/A) \geq \rho(q_I(b)/\overline{q_I(A)}).$$

Since  $q_I(A)$  is an  $\mathcal{R}_s$ -radical ideal of  $B/I$  then so is  $\overline{q_I(A)}$ . It follows from Theorem 9.13 that  $b/I$  is a point of continuity of  $\rho$ . In particular,

$$\liminf (\rho(b_n)) \geq \liminf (\rho(b_n/I)) = \rho(b/I),$$

a contradiction with (9.3).

We proved that any  $\mathcal{R}_s^p$ -radical algebra lies in  $\mathcal{V}_\rho$ . So  $\mathcal{R}_s^p \in \mathcal{F}_{\mathcal{V}_\rho}$  and  $\mathcal{R}_s^p \leq \mathcal{R}_{\vec{\rho}}$ . As  $\mathcal{R}_s^{p*}$  is the smallest radical that is larger than or equal to  $\mathcal{R}_s^p$  then  $\mathcal{R}_s^{p*} \leq \mathcal{R}_{\vec{\rho}}$ .  $\square$

**Corollary 9.17.**  $\mathcal{R}_s^a \leq \mathcal{R}_{\vec{\rho}}$ ; in particular, all  $\mathcal{R}_s^a$ -radical ideals of a Banach algebra consist of the points of continuity for  $\rho$  in the algebra.

*Proof.* It follows from Lemma 9.3 and Theorem 9.16.  $\square$

**Corollary 9.18.** Let  $A$  be a  $C^*$ -algebra,  $J = \mathcal{R}_{\text{gcr}}(A)$ , the largest GCR-ideal of  $A$ . Then the spectral radius is continuous at every point  $a \in J$ .

*Proof.* It follows from Theorem 9.11 and Theorem 9.16.  $\square$

**Problem 9.19.** Is  $\mathcal{R}_{\vec{\rho}}$  the largest radical among all radicals  $P$  for which  $P$ -radical ideals consist of the points of continuity for the spectral radius?

**Problem 9.20.** If the spectral radius is continuous on a  $C^*$ -algebra  $A$ , is  $A$  a GCR-algebra?

**9.3. Continuity of the joint spectral radius.** Let  $A$  be a normed algebra. Here it will be convenient to denote by  $\rho_j$  (instead of  $\rho$ ) the function  $M \mapsto \rho(M)$  defined on bounded sets of  $A$ . It is upper continuous with respect to Hausdorff's distance [ST1, Proposition 3.1], that is

$$\limsup_{k \rightarrow \infty} \rho(M_k) \leq \rho(M)$$

if  $M_k \rightarrow M$  in the sense that  $\text{dist}(M_k, M) \rightarrow 0$  as  $k \rightarrow \infty$ . If  $\lim_{k \rightarrow \infty} \rho(M_k) = \rho(M)$  for each sequence  $M_k \rightarrow M$  then we say that  $M$  is a *point of continuity for the joint spectral radius*.

Let  $\mathcal{V}_{\rho_j}$  be the class of all normed algebras  $A$  satisfying the following condition

- (1<sub>j</sub>) For every normed algebra  $B$  containing  $A$  as a closed ideal, any precompact set  $M \subset B$  with  $\rho(M) > \rho(M/A)$  is a point of continuity of the function  $\rho_j$ .

Let  $\mathcal{F}_{\mathcal{V}_{\rho_j}}$  be the family of all topological under radicals whose radical classes are contained in  $\mathcal{V}_{\rho_j}$ .

**Theorem 9.21.** *Let  $\mathcal{R}_{\rho_j} = \vee \mathcal{F}_{\mathcal{V}_{\rho_j}}$ . Then*

- (1)  $\mathcal{R}_{\rho_j}$  is a radical and  $\mathcal{R}_{\rho_j} \in \mathcal{F}_{\mathcal{V}_{\rho_j}}$ ;
- (2) For every normed algebra  $A$ , every precompact subset  $M$  of  $\mathcal{R}_{\rho_j}(A)$  is a point of continuity of  $\rho_j$ ;
- (3) For every normed algebra  $A$ ,  $\rho(M) = \sup \{\rho(K) : K \subset M \text{ is finite}\}$  for every precompact set  $M$  in  $\mathcal{R}_{\rho_j}(A)$ ;
- (4)  $\mathcal{R}_{\text{hc}}^r \vee \mathcal{R}_{\text{cq}}^a \leq \mathcal{R}_{\rho_j}$ .

*Proof.* (1) & (4) Using Lemma 9.14 and repeating the argument in Theorem 9.15, we obtain that  $\mathcal{R}_{\rho_j}$  is a radical and all  $\mathcal{R}_{\rho_j}$ -radical algebras lie in  $\mathcal{V}_{\rho_j}$ . It was proved in [ST7, Theorem 6.3] that for  $\mathcal{R}_{\text{hc}}^r \vee \mathcal{R}_{\text{cq}}^a$ -radical ideals the condition (1<sub>j</sub>) holds. Therefore (4) is valid.

(2) is similar to the proof of Theorem 9.15.

(3) follows from (2). □

We call  $\mathcal{R}_{\rho_j}$  the  $\rho_j$ -continuity radical.

**Corollary 9.22.** *The joint spectral radius is continuous on precompact subsets of any scattered  $C^*$ -algebra.*

In Section 10 this result will be extended to all GCR-algebras.

**9.4. Continuity of the tensor radius.** The function  $\rho_t : N \mapsto \rho_t(N)$  defined on summable families  $N = (a_m)_1^\infty$  of elements of  $A$  is upper continuous with respect to the metric  $d(N', N) = \sum_1^\infty \|a'_m - a_m\|$  where  $N' = (a'_m)_1^\infty$  (see [ST6, Proposition 3.12]). We say that a family  $N$  is a *point of continuity of the tensor spectral radius* if  $\rho_t(N_n) \rightarrow \rho_t(N)$ , for any sequence  $N_n$  of summable families in  $A$  that tends to  $N$  with respect to the metric  $d$ .

Let  $\mathcal{V}_{\rho_t}$  be the class of all normed algebras  $A$  satisfying the following condition

- (1<sub>t</sub>) For every normed algebra  $B$  containing  $A$  as a closed ideal, any summable family  $N$  in  $B$  with  $\rho_t(N) > \rho_t(N/A)$  is a point of continuity of the function  $\rho_t$ .

**Lemma 9.23.** *Let  $A$  be a normed algebra, let  $(J_\alpha)_{\alpha \in \Lambda}$  be an up-directed net of closed ideals of  $A$  and  $J = \overline{\bigcup_{\alpha \in \Lambda} J_\alpha}$ . Then*

$$\|N/J\|_+ = \lim \|N/J_\alpha\|_+ = \inf \|N/J_\alpha\|_+, \quad (9.4)$$

$$\rho_t(N/J) = \lim \rho_t(N/J_\alpha) = \inf \rho_t(N/J_\alpha) \quad (9.5)$$

for every summable family  $N = (a_m)_1^\infty$  in  $A$ .

*Proof.* As  $\|N/J\|_+ \leq \|N/J_\alpha\|_+$  then

$$\rho_t(N/J) = \inf_n \|N^n/J\|_+^{1/n} \leq \inf_\alpha \inf_n \|N^n/J_\alpha\|_+^{1/n} = \inf_\alpha \rho(N/J_\alpha)$$

and

$$\|N/J\|_+ \leq \inf_\alpha \|N/J_\alpha\|_+ \leq \liminf_\alpha \|N/J_\alpha\|_+.$$

By our assumption, for every  $a_m$  and  $\varepsilon_m > 0$  there exists  $\alpha = \alpha(a_m, \varepsilon_m)$  such that

$$\|a_m/J_\alpha\| \leq \|a_m/J\| + \varepsilon_m. \quad (9.6)$$

For  $\varepsilon > 0$ , take  $k > 0$  such that  $d(N|_1^k, N) < \varepsilon$ . Let  $N_k = N|_1^k$ . Take  $\varepsilon_m > 0$  such that  $\sum_1^k \varepsilon_m < \varepsilon$ . Then  $d(N_k/J_\gamma, N/J_\gamma) \leq d(N_k, N) < \varepsilon$ , and

$$\|N/J_\gamma\|_+ \leq d(N_k/J_\gamma, N/J_\gamma) + \|N_k/J_\gamma\|_+ \leq \|N_k/J\|_+ + \varepsilon \leq \|N/J\|_+ + \varepsilon$$

for  $\gamma \geq \max\{\alpha(a_m, \varepsilon_m) : m = 1, \dots, k\}$  by (9.6). Therefore

$$\inf_\alpha \|N/J_\alpha\|_+ \leq \limsup_\alpha \|N/J_\alpha\|_+ \leq \|N/J\|_+ \quad (9.7)$$

that implies (9.4). Take  $n > 0$  such that  $\|N^n/J\|_+^{1/n} \leq \rho(N/J) + \varepsilon$ . It follows from (9.7) applied to  $N^n$  that

$$\begin{aligned} \inf_\alpha \rho_t(N/J_\alpha) &\leq \limsup_\alpha \rho_t(N/J_\alpha) \leq \limsup_\alpha \|N^n/J_\alpha\|_+^{1/n} \leq \|N^n/J\|_+^{1/n} \\ &\leq \rho_t(N/J) + \varepsilon. \end{aligned}$$

This implies (9.5).  $\square$

**Lemma 9.24.** *Let  $A$  be a commutative Banach algebra. Then  $\rho_t$  is uniformly continuous on  $B$  with respect to the metric  $d$ .*

*Proof.* One can assume that  $A$  is unital. Let  $M = (a_n)_1^\infty$  and  $N = (b_n)_1^\infty$  be summable families in  $A$ , and let  $\mathcal{F}$  be the set of all multiplicative functionals  $f$  on  $A$  with  $\|f\| = f(1) = 1$ . By Theorem 5.29,

$$\rho_t(M) = \sup \left\{ \sum_1^\infty |f(a_n)| : f \in \mathcal{F} \right\}$$

and, by [ST6, Propositions 3.3 and 3.4],  $\rho_t$  is subadditive on  $A$ , whence

$$\begin{aligned} |\rho_t(M) - \rho_t(N)| &\leq \rho_t(M - N) = \sup \left\{ \sum_1^\infty |f(a_n - b_n)| : f \in \mathcal{F} \right\} \\ &\leq \sum_1^\infty \|a_n - b_n\| = d(M, N). \end{aligned}$$

$\square$

Let  $\mathcal{F}_{\mathcal{V}_{\rho_t}}$  be the family of all topological under radicals whose radical classes are contained in  $\mathcal{V}_{\rho_t}$ .

**Theorem 9.25.** *Let  $\mathcal{R}_{\vec{\rho}_t} = \vee \mathcal{F}_{\mathcal{V}_{\rho_t}}$ . Then*

- (1)  $\mathcal{R}_{\vec{\rho}_t}$  is a radical and  $\mathcal{R}_{\vec{\rho}_t} \in \mathcal{F}_{\mathcal{V}_{\rho_t}}$ ;
- (2) For every normed algebra  $A$ , every summable family  $N$  of  $\mathcal{R}_{\vec{\rho}_t}(A)$  is a point of continuity of  $\rho_t$ ;
- (3) For every normed algebra  $A$ ,  $\rho_t(N) = \sup_k \rho_t(N|_1^k)$  for every summable family  $N$  in  $\mathcal{R}_{\vec{\rho}_t}(A)$ ;
- (4)  $\mathcal{R}_t^a \leq \mathcal{R}_{\vec{\rho}_t}$ .

*Proof.* (1) Using Lemma 9.23 and repeating the argument in Theorem 9.15, we obtain that  $\mathcal{R}_{\vec{\rho}_t}$  is a radical and all  $\mathcal{R}_{\vec{\rho}_t}$ -radical algebras lie in  $\mathcal{V}_{\rho_t}$ .

(2) is similar to Theorem 9.15.

(3) follows from (2).

(4) It is sufficient to check the condition  $(1_t)$  for any  $\mathcal{R}_t$ -semisimple Banach algebra  $B$ . Let  $A$  be a closed  $\mathcal{R}_t^a$ -radical ideal of  $B$ . As  $A$  is  $\mathcal{R}_t$ -semisimple, it is a central ideal. By Theorem 5.19,

$$\rho_t(N) = \rho_t(L_N) \quad (9.8)$$

for every summable family  $N = (a_m)_1^\infty$  in  $B$ . As  $A$  is a closed invariant subspace for  $L_B$  then

$$\rho_t(L_N) = \max \{ \rho_t(L_N|_{B/A}), \rho_t(L_N|_A) \} \quad (9.9)$$

by Theorem 5.18. It is clear that

$$\rho_t(L_N|_{B/A}) = \rho_t(L_{N/A}) = \rho_t(N/A). \quad (9.10)$$

Let now  $\rho_t(N) > \rho_t(N/A)$ . It follows from (9.8), (9.9) and (9.10) that

$$\rho_t(N) = \rho_t(L_N|_A)$$

Let  $(N_n)$  be a sequence of summable families of  $B$  such that  $N_n \rightarrow N$  in the metric  $d$  as  $n \rightarrow \infty$ . Then  $L_{N_n}|_A \rightarrow L_N|_A$ . It is easy to check that the algebra  $L_B|_A$  is commutative. By Lemma 9.24,  $\rho_t$  is continuous on  $L_B|_A$ , whence

$$\liminf \rho_t(N_n) \geq \liminf \rho_t(L_{N_n}|_A) = \rho_t(L_N|_A) = \rho_t(N)$$

and  $\lim \rho_t(N_n) = \rho_t(N)$ .

This implies that  $\mathbf{Rad}(\mathcal{R}_t^a) \subset \mathcal{V}_{\rho_t}$ . As  $\mathcal{R}_{\vec{\rho}_t}$  is the largest radical with such property,  $\mathcal{R}_t^a \leq \mathcal{R}_{\vec{\rho}_t}$ .  $\square$

We call  $\mathcal{R}_{\vec{\rho}_t}$  the  $\rho_t$ -continuity radical.

## 10. ESTIMATIONS OF THE JOINT SPECTRAL RADIUS

In this section we study how various spectral characteristics of an element or a subset of a Banach algebra can be expressed via its images in quotients of the algebra, and how the characteristics of these images depend on the corresponding ideals. The main attention is devoted to the joint spectral radius.

For example, it is well known that for each element  $a$  of any Banach algebra  $A$

$$\rho(a) = \sup \{ \rho(a/I) : I \in \text{Prim}(A) \}. \quad (10.1)$$

Can  $\text{Prim}(A)$  be changed here by arbitrary family of (primitive or not) ideals whose intersection is contained in  $\text{Rad}(A)$ ? Is it possible to extend (10.1) to  $\rho(M)$ , where  $M$  is a precompact subset of  $A$ ? How in general depend  $\rho(a/I)$  and  $\rho(M/I)$  on  $I$ ? Questions of this type often arise in the spectral theory.

In what follows by an *operator* we mean a bounded linear operator.

**10.1. Finite families of ideals with trivial intersection.** We begin with the spectrum.

**Theorem 10.1.** *Let  $A$  be a Banach algebra,  $I_1, \dots, I_n$  be closed ideals of  $A$ , and let  $\bigcap_{k=1}^n I_k \subset \text{Rad}(A)$ . Then*

$$\sigma(a) = \bigcup_{k=1}^n \sigma(a/I_k),$$

for any element  $a \in A$ .

*Proof.* Let  $I, J$  be ideals of  $A$ . Assuming that  $0 \notin \sigma(a/I) \cup \sigma(a/J)$ , we prove that  $0 \notin \sigma(a/K)$  where  $K = I \cap J$ . Indeed, since  $a$  is invertible in  $A/I$  and  $A/J$  then there are  $b, c \in A$  with  $ba = 1 + i$  and  $ca = 1 + j$  where  $i \in I, j \in J$ . Therefore

$$(ba - 1)(ca - 1) = ij \in K.$$

It follows that  $(bac - c - b)a \in 1 + K$  that is  $a$  is left invertible in  $A/K$ . Similarly,  $a$  is right invertible in  $A/K$ . Thus  $a$  is invertible in  $A/K$ .

Now using induction, we obtain that

$$\sigma(a / \cap_{k=1}^n I_k) \subset \cup_{k=1}^n \sigma(a / I_k).$$

In our assumptions this implies that

$$\sigma(a) = \sigma(a / \text{Rad}(A)) \subset \cup_{k=1}^n \sigma(a / I_k);$$

the converse inclusion is evident.  $\square$

Theorem 10.1 implies that if  $\mathcal{F}$  is a finite set of closed ideals with intersection in  $\text{Rad}(A)$  then

$$\rho(a) = \sup_{J \in \mathcal{F}} \rho(a/J) \text{ for each } a \in A. \quad (10.2)$$

Our next aim is to establish a similar result for the joint spectral radius.

We use the following statement proved in [ST1, Corollary 4.3].

**Lemma 10.2.** *Let  $M$  be a bounded set of operators on a Banach space  $X$ , and let  $X \supset Z_1 \supset \dots \supset Z_n$  be a chain of closed subspaces invariant for all operators in  $M$ . Then*

$$\rho(M) = \max \left\{ \rho \left( M|_{X/Z_1} \right), \rho \left( M|_{Z_1/Z_2} \right), \dots, \rho \left( M|_{Z_{n-1}/Z_n} \right), \rho \left( M|_{Z_n} \right) \right\},$$

where by  $M|_{Z_k/Z_{k+1}}$  we mean the family of operators induced by  $M$  in the quotient Banach space  $Z_k/Z_{k+1}$ .

Let  $A$  be a Banach algebra. Recall (sf. Section 1.2.7) that by a *Banach ideal* of  $A$  we call any ideal  $I$  of  $A$  which is complete with respect to a norm  $\|\cdot\|_I$  such that

$$\|x\|_I \geq \|x\|$$

for all  $x \in I$ ; recall also that  $W_a := L_a R_a$ , for  $a \in A$ , and  $W_M := \{W_a : a \in M\}$ , for  $M \subset A$ . It is clear that all ideals of  $A$  are invariant subspaces for the operators  $W_a$ .

Let  $\rho_{(A; \|\cdot\|_I)}(M)$  denote the joint spectral radius of a bounded set  $M$  in the algebra  $(A; \|\cdot\|_I)$ .

**Lemma 10.3.** *Let  $A$  be a Banach algebra, and let  $I$  be a Banach ideal of  $A$ . Then*

- (1) *All operators  $W_a|_I$ ,  $a \in A$ , are bounded in the norm  $\|\cdot\|_I$ ;*
- (2) *For each bounded subset  $M$  of  $A$ ,*

$$\rho_{(I; \|\cdot\|_I)}(W_M|_I) \leq \rho_{(A; \|\cdot\|_I)}(M)^2. \quad (10.3)$$

*Proof.* (1) It was proved in [B] that there is a constant  $C > 0$  such that

$$\|axb\|_I \leq C\|a\|_A\|x\|_I\|b\|_A$$

for all  $a, b \in A^1, x \in I$ . Therefore  $\|W_a|_I\|_I \leq C\|a\|^2$  which proves (1).

(2) As  $\|W_M|_I\|_I \leq C(\|M^n\|_A)^2$  then  $\|W_M|_I\|_I^{1/n} \leq C^{1/n}(\|M^n\|_A)^{2/n}$ , and taking  $n \rightarrow \infty$  we obtain (2).  $\square$

**Proposition 10.4.** *Let  $A$  be a Banach algebra, let  $I_1, \dots, I_n$  be closed ideals of  $A$ , and let  $J = \cap_{i=1}^n I_i$ . Then*

$$\rho(M)^2 = \max \left\{ \max_{1 \leq i \leq n} \rho(M/I_i)^2, \rho(W_M|_J) \right\}, \quad (10.4)$$

for each bounded subset  $M$  of  $A$ .

*Proof.* We will prove  $\leq$  because the converse inequality is trivial.

For  $k \leq n$ , let  $J_k = I_1 \cap I_2 \cap \dots \cap I_k$ . Applying Lemma 10.2 to the family  $W = W_M$  of operators on  $A$  and the chain  $A \supset J_1 \supset J_2 \dots \supset J_n$ , we obtain that

$$\begin{aligned} \rho(M)^2 &= \rho(W_M) \\ &= \max \{ \rho(W_M|_{A/J_1}), \rho(W_M|_{J_1/J_2}), \dots, \rho(W_M|_{J_{n-1}/J_n}), \rho(W_M|_{J_n}) \}. \end{aligned}$$

So it suffices to show that

$$\rho(W_M|_{J_{k-1}/J_k}) \leq \rho(M/I_k)^2 \quad (10.5)$$

for each  $k$  (assuming  $J_0 = A$ ).

For  $k = 1$ , the inequality (10.5) is in fact an equality:

$$\rho(W_M|_{A/I_1}) = \rho(M/I_1)^2.$$

For a fixed  $k$  with  $1 < k \leq n$ , let  $B = A/I_k$  and  $q = q_{I_k} : A \rightarrow B$ . Then the algebraic isomorphism  $\phi$  of  $I := q(J_{k-1}) = (J_{k-1} + I_k)/I_k$  onto the Banach algebra  $C := J_{k-1}/(I_k \cap J_{k-1}) = J_{k-1}/J_k$  allows us to supply  $I$  with a new norm

$$\|x\|_I = \|\phi(x)\|_C,$$

and it is easy to check that  $I$  is a Banach ideal of  $B$  in this norm. It follows from the definition of the norm  $\|\cdot\|_I$  that  $\|W_a|_{J_{k-1}/J_k}\| = \|W_{q(a)}|_I\|_I$ , for  $a \in A$ . Therefore

$$\rho(W_M|_{J_{k-1}/J_k}) = \rho_{(I, \|\cdot\|_I)}(W_{q(M)}|_I). \quad (10.6)$$

Applying Lemma 10.3 to the subset  $q(M)$  of  $B$  we get:

$$\rho_{(I, \|\cdot\|_I)}(W_{q(M)}|_I) \leq \rho_B(q(M))^2 = \rho(M/I_k)^2.$$

By (10.6), this is a reformulation of (10.5).  $\square$

**Corollary 10.5.** *Let  $A$  be a Banach algebra, let  $\mathcal{F} = \{I_1, \dots, I_n\}$  be a finite family of closed ideals of  $A$ , such that  $\cap_{i=1}^n I_i \subset \mathcal{R}_{\text{cq}}(A)$ . Then*

$$\rho(M) = \max \{ \rho(M/I_1), \dots, \rho(M/I_n) \}, \quad (10.7)$$

for each precompact subset  $M$  of  $A$ .

*Proof.* The equality (10.7) is a consequence of Theorem 10.4, if  $\cap_{i=1}^n I_i = 0$ . It follows that in general

$$\rho(M/(\cap_{i=1}^n I_i)) = \max \{ \rho(M/I_1), \dots, \rho(M/I_n) \}. \quad (10.8)$$

Indeed, setting  $J = \cap_{i=1}^n I_i$ ,  $J_i = I_i/J$  and  $\widetilde{M} = M/J$ , we have that

$$\cap_{i=1}^n J_i = 0$$

whence  $\rho(\widetilde{M}) = \max_i \rho(\widetilde{M}/J_i)$ , and it suffices to note that  $\widetilde{M}/J_i$  corresponds to  $M/I_i$  with respect to the standard isomorphism of  $A/I_i$  onto  $(A/J)/J_i$ .

Now, since  $\cap_{i=1}^n I_i \subset \mathcal{R}_{\text{cq}}(A)$  and (2.7) holds for precompact sets, we obtain that

$$\rho(M) = \rho(M/\mathcal{R}_{\text{cq}}(A)) \leq \rho(M/(\cap_{i=1}^n I_i)) = \max \{ \rho(M/I_1), \dots, \rho(M/I_n) \}.$$

The converse inequality is evident.  $\square$



**10.2. Arbitrary families of ideals with trivial intersection.** The result of Theorem 10.1 does not extend to arbitrary families of ideals. It suffices to show that the equality (10.2) fails in general.

**Example 10.6.** Let  $\{e_n : 1 \leq n < \infty\}$  be an orthonormal basis in a Hilbert space  $H$ , and let  $A$  be the algebra of all operators on  $H$  preserving the subspaces  $H_n = \text{span}(e_1, \dots, e_n)$ . Let  $K_n = \{T \in A : T|_{H_n} = 0\}$ . Then all  $K_n$  are closed ideals of  $A$ , and

$$\bigcap_{n=1}^{\infty} K_n = 0.$$

Let  $S$  be the backward shift:  $Se_n = e_{n-1}$ ,  $Se_1 = 0$ . Then  $S \in A$  and all elements  $S/K_n$  are nilpotent,

$$\rho(S/K_n) = 0$$

while  $\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$  and

$$\rho(S) = 1.$$

Let us first of all give a slight extension of Theorem 10.1.

**Proposition 10.7.** Let  $A$  be an algebra, and let  $\mathcal{F} = (I_\alpha)_{\alpha \in \Lambda}$  be a family of ideals of  $A$ . For each  $\alpha \in \Lambda$ , set  $J_\alpha = \bigcap_{\beta \neq \alpha} I_\beta$ . If  $\bigcap_{\alpha \in \Lambda} I_\alpha \subset \text{rad}(A)$  then

$$\sigma(a) = (\bigcup_{\alpha \in \Lambda} \sigma(a/I_\alpha)) \cup (\bigcap_{\alpha \in \Lambda} \sigma(a/J_\alpha)),$$

for each  $a \in A$

*Proof.* For each  $\alpha \in \Lambda$ ,  $I_\alpha \cap J_\alpha \subset \text{rad}(A)$ , whence

$$\sigma(a) = \sigma(a/I_\alpha) \cup \sigma(a/J_\alpha)$$

by Theorem 10.1. Now if  $\lambda \in \sigma(a)$  and  $\lambda \notin \bigcup_{\alpha \in \Lambda} \sigma(a/I_\alpha)$  then  $\lambda \in \sigma(a/J_\alpha)$  for every  $\alpha$ . Therefore  $\lambda \in \bigcap_{\alpha \in \Lambda} \sigma(a/J_\alpha)$ .  $\square$

The following result shows that for scattered Banach algebras the situation is sufficiently satisfactory.

**Theorem 10.8.** Let  $A$  be a Banach algebra. If the spectrum of an element  $a$  of  $A$  is the closure of its isolated points then  $\sigma(a) = \overline{\bigcup_{\alpha \in \Lambda} \sigma(a/I_\alpha)}$  for each family  $\mathcal{F} = (I_\alpha)_{\alpha \in \Lambda}$  of closed ideals with  $\bigcap_{\alpha \in \Lambda} I_\alpha \subset \text{Rad}(A)$ .

*Proof.* It suffices to show that  $\lambda \in \bigcup_{\alpha \in \Lambda} \sigma(a/I_\alpha)$  for each isolated point  $\lambda \in \sigma(a)$ . Assume, to the contrary, that

$$\lambda \notin \sigma(a/I_\alpha)$$

for each  $\alpha$ . Let  $p$  be the Riesz projection of  $a$  corresponding to  $\lambda$ . Then  $(a - \lambda)p$  is quasinilpotent whence  $(a - \lambda)p/I_\alpha$  is quasinilpotent for each  $\alpha$ . But  $(a - \lambda)/I_\alpha$  is an invertible element of the algebra  $A/I_\alpha$  that commutes with  $p/I_\alpha$ . Therefore  $p/I_\alpha$  is quasinilpotent. As it is idempotent,  $p/I_\alpha = 0$  whence  $p \in I_\alpha$ . Then

$$p \in \bigcap_{\alpha \in \Lambda} I_\alpha \subset \text{Rad}(A),$$

a contradiction.  $\square$

Let us now come to consideration of the joint spectral characteristics of subsets in  $A$ . Recall that the *Berger-Wang spectral radius*  $r(M)$  of a bounded family  $M \subset A$  is defined by

$$r(M) = \limsup_{n \rightarrow \infty} \sup_{a \in M^n} \rho(a)^{1/n}.$$

**Proposition 10.9.** *Let  $A$  be a Banach algebra, and let  $\mathcal{F}$  be a family of closed ideals such that*

$$\rho(a) = \sup_{J \in \mathcal{F}} \rho(a/J)$$

*for each  $a \in A$ . Then*

$$r(M) = \sup_{I \in \mathcal{F}} r(M/I) \quad (10.9)$$

*for each bounded set  $M \subset A$ .*

*Proof.* By [ST7, (2.1)],

$$r(M) = \sup_n \sup_{a \in M^n} \rho(a)^{1/n}.$$

Hence by (10.2),

$$r(M) = \sup_n \sup_{a \in M^n} \sup_{I \in \mathcal{F}} \rho(a/I)^{1/n} = \sup_{I \in \mathcal{F}} \sup_n \sup_{a \in M^n} \rho(a/I)^{1/n} = \sup_{I \in \mathcal{F}} r(M/I).$$

□

**Corollary 10.10.** *Let  $A$  be a Banach algebra, and let  $\mathcal{F}$  be a finite family of closed ideals. If  $\cap\{I : I \in \mathcal{F}\} \subset \text{Rad}(A)$  then*

$$r(M) = \sup_{I \in \mathcal{F}} r(M/I)$$

*for each bounded set  $M \subset A$ .*

Now we consider the joint spectral radius  $\rho(M)$ .

**Proposition 10.11.** *Let  $A$  be a Banach algebra, let  $\mathcal{F}$  be a family of closed ideals of  $A$ , and let  $\mathcal{E}$  be the set of all finite intersections of ideals from  $\mathcal{F}$ . Then*

$$\rho(M)^2 = \max \left\{ \max_{I \in \mathcal{F}} \rho(M/I)^2, \inf_{J \in \mathcal{E}} \rho(W_M|_J) \right\}$$

*for every bounded set  $M$  in  $A$ .*

*Proof.* Only the inequality  $\leq$  should be proved. If  $\inf_{J \in \mathcal{E}} \rho(W_M|_J) < \rho(M)^2$  then there are ideals  $I_1, \dots, I_n \in \mathcal{F}$  with

$$\rho(W_M|_J) < \rho(M)^2$$

where  $J = I_1 \cap I_2 \cap \dots \cap I_n$ . By Proposition 10.4,

$$\rho(M)^2 = \max \left\{ \max_{1 \leq i \leq n} \rho(M/I_i)^2, \rho(W_M|_J) \right\} = \max_{1 \leq i \leq n} \rho(M/I_i)^2 \leq \max_{I \in \mathcal{F}} \rho(M/I)^2$$

and we are done. □

Now we shall estimate the number  $\inf_{J \in \mathcal{E}} \rho(W_M|_J)$ . Let  $\mathcal{E}$  be a family of closed subspaces of a Banach space  $X$ , and let

$$\begin{aligned} \text{Alg}(\mathcal{E}) &= \{T \in \mathcal{B}(X) : TY \subset Y \text{ for all } Y \in \mathcal{E}\}, \\ \|T\|_{\mathcal{E}} &= \inf\{\|T|_Y\| : Y \in \mathcal{E}\} \text{ for any } T \in \text{Alg}(\mathcal{E}), \\ \ker \mathcal{E} &= \{T \in \text{Alg}(\mathcal{E}) : \|T\|_{\mathcal{E}} = 0\}. \end{aligned}$$

For a bounded set  $N \subset \text{Alg}(\mathcal{E})$ , we write

$$\|N\|_{\mathcal{E}} = \sup_{T \in N} \|T\|_{\mathcal{E}}.$$

**Lemma 10.12.** *Let  $X$  be a Banach space, and let  $\mathcal{E}$  be a family of closed subspaces of  $X$ , closed under finite intersections and satisfying the condition  $\cap_{Y \in \mathcal{E}} Y = 0$ . Then*

- (1<sub>e</sub>)  $\|\cdot\|_{\mathcal{E}}$  is a seminorm on  $\text{Alg}(\mathcal{E})$ ;
- (2<sub>e</sub>)  $\|TS\|_{\mathcal{E}} \leq \|T\|_{\mathcal{E}} \|S\|_{\mathcal{E}}$  for any  $T, S \in \text{Alg}(\mathcal{E})$ ;

- (3<sub>e</sub>)  $\|T\|_{\mathcal{E}} \leq \|T\|$  for all  $T \in \text{Alg}(\mathcal{E})$ ;  
 (4<sub>e</sub>)  $\|T\|_{\mathcal{E}} = 0$  for each operator  $T \in \text{Alg}(\mathcal{E})$  which is compact on some  $Y \in \mathcal{E}$ .

*Proof.* (1<sub>e</sub>) and (2<sub>e</sub>) follow from the fact that  $\|T|_{Y_1 \cap Y_2}\| \leq \min\{\|T|_{Y_1}\|, \|T|_{Y_2}\|\}$ , (3<sub>e</sub>) is evident.

(4<sub>e</sub>) Let  $T \in \text{Alg}(\mathcal{E})$  be compact on  $Y \in \mathcal{E}$ . Let  $Z_{\odot}$  be the closed unit ball of  $Z \in \mathcal{E}$ , and  $K_Z = \overline{T(Z_{\odot})}$ . If  $\|T\|_{\mathcal{E}} > 0$ , choose a number  $t$  such that  $0 < t < \|T\|_{\mathcal{E}}$ . For each  $Z \in \mathcal{E}$ , let  $D_Z = \{\zeta \in K_Z : \|\zeta\| \geq t\}$ . Then  $(D_{Z \cap Y})_{Z \in \mathcal{E}}$  is a centered family of compact sets, so there is an element  $z$  in their intersection. It is non-zero and belongs to  $\cap\{Y \cap Z : Z \in \mathcal{E}\} = 0$ , a contradiction.  $\square$

It follows from (3<sub>e</sub>) that  $\|\cdot\|_{\mathcal{E}}$  is continuous with respect to  $\|\cdot\|$  on  $\text{Alg}(\mathcal{E})$ . This together with (2<sub>e</sub>) yield that  $\ker \mathcal{E}$  is a closed ideal of  $\text{Alg}(\mathcal{E})$ .

We return to the conditions of Proposition 10.11. Let  $A$  be a Banach algebra, and let  $J$  be a closed ideal of  $A$ . Let  $\mathbb{K}_J(A)$  be the set of all closed ideals  $I$  of  $A$  such that  $I$  is generated by a compact element of  $J$ . (Recall that  $a$  is a *compact element* of  $A$  if  $W_a$  is a compact operator on  $A$ .) Set

$$\mathbb{K}_{\mathcal{E}}(A) = \cup_{J \in \mathcal{E}} \mathbb{K}_J(A).$$

**Lemma 10.13.** *Let  $A$  be a Banach algebra, let  $\mathcal{F}$  be a family of closed ideals of  $A$  with zero intersection, and let  $\mathcal{E}$  be the set of all finite intersections of ideals from  $\mathcal{F}$ . If  $K \in \mathbb{K}_{\mathcal{E}}(A)$  then  $L_K R_K \subset \ker \mathcal{E}$  and*

$$\inf_{J \in \mathcal{E}} \rho(W_M|_J) \leq \rho(M) \inf_{K \in \mathbb{K}_{\mathcal{E}}(A)} \rho(M/K)$$

for every bounded set  $M$  in  $A$ .

*Proof.* Let  $L \in \mathcal{E}$ , let  $a$  be a compact element of  $L$ ,  $I$  the ideal of  $A$  generated by  $a$ , and  $K = \overline{I}$ . Then  $W_a \in \text{Alg}(\mathcal{E})$  and it is compact on  $L$ . By Lemma 10.12,  $\|W_a\|_{\mathcal{E}} = 0$ . Note that the operator  $L_x R_y$  for any  $x, y \in I$  is represented as a finite sum  $\sum L_{u_i} R_{v_i} W_a L_{s_i} R_{t_i}$  for some  $u_i, v_i, s_i, t_i \in A^1$  whence

$$\|L_x R_y\|_{\mathcal{E}} = \left\| \sum L_{u_i} R_{v_i} W_a L_{s_i} R_{t_i} \right\|_{\mathcal{E}} \leq \sum \|L_{u_i} R_{v_i}\|_{\mathcal{E}} \|W_a\|_{\mathcal{E}} \|L_{s_i} R_{t_i}\|_{\mathcal{E}} = 0.$$

So  $L_K R_K \subset \ker \mathcal{E}$ .

Let  $M$  be a bounded set in  $A$ . Denote  $\|M/K\|$  by  $d$ . For  $a, b \in M$  and  $\varepsilon > 0$  choose  $u, v \in K$  with

$$\|a - u\| \leq d + \varepsilon \text{ and } \|b - v\| \leq d + \varepsilon.$$

Then  $\|L_u R_v\|_{\mathcal{E}} = 0$  whence

$$\begin{aligned} \|L_a R_b\|_{\mathcal{E}} &= \|L_u R_v + L_u R_{b-v} + L_{a-u} R_v + L_{a-u} R_{b-v}\|_{\mathcal{E}} \\ &\leq \|L_u R_{b-v}\|_{\mathcal{E}} + \|L_{a-u} R_v\|_{\mathcal{E}} + \|L_{a-u} R_{b-v}\|_{\mathcal{E}} \\ &\leq \|u\|(d + \varepsilon) + \|v\|(d + \varepsilon) + (d + \varepsilon)^2 \\ &\leq (\|a\| + d + \varepsilon)(d + \varepsilon) + (\|b\| + d + \varepsilon)(d + \varepsilon) + (d + \varepsilon)^2. \end{aligned}$$

Taking  $\varepsilon \rightarrow 0$  we get that  $\|L_a R_b\|_{\mathcal{E}} \leq d(\|a\| + \|b\| + 3d) \leq 5\|M/K\|\|M\|$ . Therefore

$$\|W_M\|_{\mathcal{E}} \leq 5\|M/K\|\|M\|. \quad (10.10)$$

Changing  $M$  by  $M^n$  in (10.10) we have  $\|W_{M^n}\|_{\mathcal{E}} \leq 5\|M^n/K\|\|M^n\|$ . Therefore, there is  $J \in \mathcal{E}$  with

$$\|W_{M^n}|_J\| \leq 6\|M^n/K\|\|M^n\|$$

whence  $\rho(W_M|_J) \leq \|W_{M^n}|_J\|^{1/n} \leq 6^{1/n} \|M^n/K\|^{1/n} \|M^n\|^{1/n}$ . Thus

$$\inf_{J \in \mathcal{E}} \rho(W_M|_J) \leq \inf_n \left( 6^{1/n} \|M^n/K\|^{1/n} \|M^n\|^{1/n} \right) = \rho(M/K) \rho(M)$$

for any  $K \in \mathbb{K}(A)$ . This is what we need.  $\square$

**Theorem 10.14.** *Let  $A$  be a Banach algebra, let  $\mathcal{F}$  be a family of closed ideals of  $A$  such that  $\bigcap_{I \in \mathcal{F}} I = 0$ , and let  $\mathcal{E}$  be the set of all finite intersections of ideals from  $\mathcal{F}$ . Then, for any bounded subset  $M$  of  $A$ ,*

$$\rho(M) = \max \left\{ \inf_{K \in \mathbb{K}_{\mathcal{E}}(A)} \rho(M/K), \max_{I \in \mathcal{F}} \rho(M/I) \right\}. \quad (10.11)$$

*Proof.* Using the result of Lemma 10.13, we get from Proposition 10.11 that

$$\begin{aligned} \rho(M)^2 &\leq \max \left\{ \rho(M) \inf_{K \in \mathbb{K}_{\mathcal{E}}(A)} \rho(M/K), \max_{I \in \mathcal{F}} \rho(M/I)^2 \right\} \\ &\leq \rho(M) \max \left\{ \inf_{K \in \mathbb{K}_{\mathcal{E}}(A)} \rho(M/K), \max_{I \in \mathcal{F}} \rho(M/I) \right\} \end{aligned}$$

whence we obtain the inequality  $\leq$ . The opposite inequality is trivial.  $\square$

**10.3. The joint spectral radius and primitive ideals.** We already mentioned and used the fact that the equality (10.2) holds for  $\mathcal{F} = \text{Prim}(A)$ . Since  $\bigcap \{I : I \in \text{Prim}(A)\} = \text{Rad}(A)$ , it follows from Proposition 10.9 that

$$r(M) = \sup_{I \in \text{Prim}(A)} r(M/I) \text{ for each bounded } M \subset A. \quad (10.12)$$

Since  $\rho(\pi(a)) \leq \rho(a/\ker \pi)$  the following well known equality extends (10.2), for the case  $\mathcal{F} = \text{Prim}(A)$ :

$$\rho(a) = \sup_{\pi \in \text{Irr}(A)} \rho(\pi(a)), \text{ for each } a \in A. \quad (10.13)$$

Arguing as in the proof of Proposition 10.9 we obtain a more strong version of (10.12):

**Proposition 10.15.** *Let  $A$  be a Banach algebra. Then*

$$r(M) = \sup_{\pi \in \text{Irr}(A)} r(\pi(M))$$

*for each bounded subset  $M$  of  $A$ .*

We are looking for the conditions that provide the validity of similar statements for the joint spectral radius:

$$\rho(M) = \sup_{I \in \text{Prim}(A)} \rho(M/I) \text{ for each precompact } M \subset A, \quad (10.14)$$

$$\rho(M) = \sup_{\pi \in \text{Irr}(A)} \rho(\pi(M)) \text{ for each precompact } M \subset A. \quad (10.15)$$

Recall that a bounded subset  $M$  of a normed algebra  $A$  is a *point of continuity* of the joint spectral radius if  $\rho(M_n) \rightarrow \rho(M)$ , for each sequence of bounded subsets  $M_n \subset A$  that tends to  $M$  in the Hausdorff metric.

**Lemma 10.16.** *Let  $A$  be a normed algebra, and let  $M$  be a bounded set of  $A$  such that  $\rho(M) = r(M)$ . If  $M$  consists of the points of continuity of the (usual) spectral radius then  $M$  is a point of continuity of the joint spectral radius.*

*Proof.* Let  $M_n \rightarrow M$ . Since the joint spectral radius is upper semicontinuous [ST1, Proposition 3.1], we have only to show that  $\liminf_{n \rightarrow \infty} \rho(M_n) \geq \rho(M)$ .

Assume, to the contrary, that

$$\lim_{n \rightarrow \infty} \rho(M_n) < 1 < \rho(M).$$

Since  $\rho(M) = r(M)$ , there are  $k \in \mathbb{N}$  and  $T \in M^k$  such that  $\rho(T) > 1$ . Clearly there are  $T_n \in M_n^k$  with  $T_n \rightarrow T$ ; since  $\rho$  is continuous at  $T$  then  $\rho(T_n) \rightarrow \rho(T)$ . But

$$\rho(T_n) \leq \rho(M_n^k) = \rho(M_n)^k < 1$$

for sufficiently big  $n$ , a contradiction.  $\square$

Recall that a normed algebra  $A$  is a *Berger-Wang algebra* if  $\rho(M) = r(M)$ , for each precompact subset  $M$  of  $A$ . It follows immediately from Proposition 10.15 that (10.14) and (10.15) hold for every Berger-Wang algebra.

**Theorem 10.17.** *Let  $A$  be a Berger-Wang Banach algebra. Then every precompact subset of  $\mathcal{R}_s^{p*}(A)$  is a point of continuity of the joint spectral radius.*

*Proof.* Follows from Theorem 9.16 and Lemma 10.16.  $\square$

Recall that a non-necessarily Hausdorff topological space  $T$  is called *quasicompact* if each its open covering contains a finite subcovering. A function  $\phi : T \rightarrow \mathbb{R}$  is *lower (upper) semicontinuous* if for each  $\lambda \in \mathbb{R}$ , the set  $\{t \in T : \phi(t) \leq \lambda\}$  (respectively  $\{t \in T : \phi(t) \geq \lambda\}$ ) is closed.

The following result is a variation of the classical Dini Theorem:

**Theorem 10.18.** *Let  $f_n$  be a decreasing sequence of functions on a quasicompact space  $T$  pointwise converging to a function  $f$ . If all  $f_n$  are upper semicontinuous then*

$$\sup_{t \in T} f_n(t) \rightarrow \sup_{t \in T} f(t) \text{ as } n \rightarrow \infty.$$

*Proof.* Choose a number  $d > \sup_{t \in T} f(t)$ . If  $\sup_{t \in T} f_n(t) > d$ , for all  $n$ , then the sets  $E_n = \{t \in T : f_n(t) \geq d\}$  are non-empty and closed. Since  $E_n \subset E_{n-1}$  and  $T$  is quasicompact, there is a point  $t_0 \in \bigcap_n E_n$ . Thus  $f_n(t_0) \geq d$  for all  $n$ , and therefore

$$f(t_0) = \lim_{n \rightarrow \infty} f_n(t_0) \geq d > \sup_{t \in T} f(t),$$

a contradiction.

It follows that

$$\lim_{n \rightarrow \infty} \sup_{t \in T} f_n(t) = \inf_{n \rightarrow \infty} \sup_{t \in T} f_n(t) \leq \sup_{t \in T} f(t).$$

On the other hand,

$$\lim_{n \rightarrow \infty} \sup_{t \in T} f_n(t) \geq \sup_{t \in T} f(t)$$

because  $f_n(t) \geq f(t)$  for all  $t$ .  $\square$

We consider the following properties which a Banach algebra  $A$  can have:

- (1<sub>c</sub>) For each  $a \in A$ ,  $\|a\| = \sup\{\|a/I\| : I \in \text{Prim}(A)\}$ ;
- (2<sub>c</sub>) For each  $a \in A$ , the map  $I \mapsto \|a/I\|$  is upper semicontinuous on  $\text{Prim}(A)$ .

**Lemma 10.19.** *Let  $A$  be a Banach algebra, and let  $M$  be a precompact subset of  $A$ . Then*

- (1) *If  $A$  has the property (1<sub>c</sub>) then  $\|M\| = \sup_{I \in \text{Prim}(A)} \|M/I\|$ ;*
- (2) *If  $A$  has the property (2<sub>c</sub>) then the map  $I \mapsto \|M/I\|$  is upper semicontinuous on  $\text{Prim}(A)$ .*

*Proof.* (1) For each  $\varepsilon > 0$ , choose  $a \in M$  with  $\|a\| > \|M\| - \varepsilon$ . Using (1<sub>c</sub>), choose  $I_0 \in \text{Prim}(A)$  with  $\|a/I_0\| > \|a\| - \varepsilon$ . Then

$$\|M\| < \|a/I_0\| + 2\varepsilon \leq \|M/I_0\| + 2\varepsilon \leq \sup_I \|M/I\| + 2\varepsilon.$$

Taking  $\varepsilon \rightarrow 0$  we get that  $\|M\| \leq \sup_I \|M/I\|$ , the converse is evident.

(2) If  $N \subset M$  is finite then  $I \mapsto \|N/I\|$  is upper semicontinuous because the set

$$\{I \in \text{Prim}(A) : \|N/I\| \geq \lambda\} = \bigcup_{a \in N} \{I \in \text{Prim}(A) : \|a/I\| \geq \lambda\}$$

is closed being a finite union of closed sets. Now, for each  $\varepsilon > 0$ , let  $N_\varepsilon$  be a finite  $\varepsilon$ -net in  $M$ . Then  $E_\varepsilon := \{I \in \text{Prim}(A) : \|N_\varepsilon/I\| \geq \lambda - \varepsilon\}$  is closed and therefore

$$\{I \in \text{Prim}(A) : \|M/I\| \geq \lambda\} = \bigcap_{\varepsilon > 0} E_\varepsilon$$

is closed.  $\square$

**Theorem 10.20.** *Let  $A$  be a Banach algebra satisfying  $(1_c)$  and  $(2_c)$ , and let  $M$  be a precompact subset of  $A$ . If  $\rho(M/I) = r(M/I)$  for all  $I \in \text{Prim}(A)$ , then*

$$\rho(M) = \sup_{I \in \text{Prim}(A)} \rho(M/I) = r(M). \quad (10.16)$$

*Proof.* The functions  $I \mapsto f_n(I) := \|M^{2^n}/I\|^{1/2^n}$  are upper semicontinuous by Lemma 10.19(2). Moreover, they decrease ( $f_{n+1}(I) \leq f_n(I)$ ) and

$$\lim_n f_n(I) = \rho(M/I) = r(M/I).$$

Thus by Theorem 10.18,  $\sup_I f_n(I) \rightarrow \sup_I \rho(M/I)$ . Using Lemma 10.19(1), one gets

$$\sup_I f_n(I) = \sup_I \|M^{2^n}/I\|^{1/2^n} = \|M^{2^n}\|^{1/2^n} \rightarrow \rho(M).$$

We proved that

$$\rho(M) = \sup_{I \in \text{Prim}(A)} \rho(M/I);$$

the second equality follows from Theorem 10.15.  $\square$

It is convenient to formulate an analogue of Theorem 10.20 for algebras of vector-valued functions on arbitrary compacts.

**Theorem 10.21.** *Let  $B$  be a Berger-Wang Banach algebra, and  $A = C(T, B)$ , the algebra of all continuous  $B$ -valued functions from a quasicompact space  $T$  to  $B$ , supplied with the sup-norm. Then  $A$  is a Berger-Wang algebra.*

*If the spectral radius function  $x \mapsto \rho(x)$  is continuous on  $B$  then the same is true for  $A$ , and moreover, the joint spectral radius is continuous on  $B$ .*

*Proof.* We argue as in the proof of Theorem 10.20 with the change of  $\text{Prim}(A)$  by  $T$ . The analogue of  $(1_c)$  follows from the definition of the norm in  $A$ . The analogue of  $(2_c)$  holds by definition (functions are continuous, so the norms are continuous).

Let  $M$  be a precompact subset of  $A$ ; for each  $t \in T$ , let  $M(t) = \{a(t) : a \in M\}$ . Arguing as above, we prove that

$$\rho(M) = \sup_{t \in T} \rho(M(t)) = r(M). \quad (10.17)$$

Let  $x \mapsto \rho(x)$  be continuous on  $B$ . To check the continuity of  $\rho$  on  $A$ , note that for  $a \in A$

$$\rho(a) = \sup_{t \in T} \rho(a(t))$$

(a special case of (10.17)). Now if  $a_\lambda \rightarrow a$  in  $A$  then  $a_\lambda(t) \rightarrow a(t)$  in  $B$ , for each  $t \in T$ , whence

$$\rho(a(t)) = \lim_\lambda \rho(a_\lambda(t)) \leq \liminf_\lambda \rho(a_\lambda).$$

Therefore  $\rho(a) \leq \liminf_\lambda \rho(a_\lambda)$ . Since upper semicontinuity holds in general, we get the continuity of  $a \mapsto \rho(a)$  on  $A$ .

The continuity of the joint spectral radius follows from Lemma 10.16.  $\square$

One can take for  $B$  an arbitrary algebra of compact operators on a Banach space  $X$  (for this case, the equality  $\rho(M) = r(M)$  was established in [ST1]).

**10.4. A C\*-algebra version of the joint spectral radius formula.** In [ST7] it was shown that for each Banach algebra  $A$  and each precompact set  $M \subset A$ ,

$$\rho(M) = \max\{r(M), \rho(M/\mathcal{R}_{\text{hc}}(A))\}. \quad (10.18)$$

It is interesting and important question if dealing with C\*-algebras one can change in (10.18) the ideal  $\mathcal{R}_{\text{hc}}(A)$  by the much larger ideal  $\mathcal{R}_{\text{gcr}}(A)$ . In other words, we study the validity of the equality

$$\rho(M) = \max\{r(M), \rho(M/\mathcal{R}_{\text{gcr}}(A))\}. \quad (10.19)$$

Our approach is based on the consideration of primitive ideals; in particular we study C\*-algebras for which the equality (10.16) holds.

The results of previous sections give some valuable information on this question. Note that the condition  $(1_c)$  holds for each C\*-algebra. The condition  $(2_c)$  holds for C\*-algebras with Hausdorff spectra. Indeed, it is proved in [Dm, Proposition 3.3.7] that for each element  $a \in A$ , the set  $\{I \in \text{Prim}(A) : \|a/I\| \geq \lambda\}$  is quasi-compact; thus if  $\text{Prim}(A)$  is Hausdorff then this set is closed, whence the function  $I \mapsto \|a/I\|$  is upper semicontinuous (in fact, continuous because it is always lower semicontinuous [Dm, Proposition 3.3.2]). Thus we obtain from Theorem 10.20 that the equality (10.16) holds for each CCR C\*-algebra  $A$  with Hausdorff  $\text{Prim}(A)$ .

We will obtain much more general results here. Let us say that a topological space has the *property (QC)* if the intersection of any down-directed net of non-empty quasicompact subsets is non-empty. Of course, each Hausdorff space has this property.

**Lemma 10.22.** *Let  $T$  be a topological space. Then*

- (1) *If  $T = \cup_{\lambda \in \Lambda} U_\lambda$ , where the net  $\{U_\lambda : \lambda \in \Lambda\}$  is up-directed, all  $U_\lambda$  are open and have property (QC), then  $T$  has property (QC);*
- (2) *If  $T = F \cup U$ , where  $F$  is closed and Hausdorff,  $U$  is open and has property (QC), then  $T$  has property (QC);*
- (3) *If  $T$  has the property (QC) then each subset of  $T$  has this property.*

*Proof.* Let  $\{E_\omega : \omega \in \Omega\}$  be a down-directed set of quasicompact subsets.

(1) For some  $\omega_1 \in \Omega$ , let  $\lambda_1, \dots, \lambda_n \in \Lambda$  be such that  $E_{\omega_1} \subset \cup_{k=1}^n U_{\lambda_k}$ . Since the net  $\{U_\lambda : \lambda \in \Lambda\}$  is up-directed,

$$E_{\omega_1} \subset U_{\lambda_0}$$

for some  $\lambda_0 \in \Lambda$ . Then all  $E_\omega$  with  $\omega > \omega_1$  are contained in the space  $U_{\lambda_0}$  which has the property (QC). By definition, their intersection is non-empty whence  $\cap_{\omega \in \Omega} E_\omega$  is non-empty.

(2) Let us check firstly that the intersection of a quasicompact set  $K$  with a closed set  $W$  is quasicompact. Indeed, if  $\{U_\lambda : \lambda \in \Lambda\}$  is a family of open subsets in  $T$  with  $K \cap W \subset \cup_{\lambda \in \Lambda} U_\lambda$  then

$$K \subset (\cup_{\lambda \in \Lambda} U_\lambda) \cup (T \setminus W);$$

choosing a finite subcovering and removing  $T \setminus W$  we obtain a finite subcovering of  $K \cap W$ .

Now in assumptions of (2) all sets  $E_\omega \cap F$  are quasicompact subsets of a Hausdorff space, so they are compact. Hence if all  $E_\omega \cap F$  are non-empty then they have non-empty intersection.

On the other hand, if  $E_{\omega_0} \cap F = \emptyset$ , for some  $\omega_0 \in \Omega$ , then  $E_{\omega_0} \subset U$ , and the same is true for all  $\omega > \omega_0$ , so the intersection is non-empty because  $U$  has property (QC).

(3) Let  $W$  be an arbitrary subset of  $T$ . It follows easily from the definition that if  $E \subset W$  is quasicompact in  $W$  then it is quasicompact in  $T$ . Thus a down-directed

net of quasicompact subsets of  $W$  is a down-directed net of quasicompact subsets of  $T$ ; by the assumptions, the intersection is non-void.  $\square$

Let  $\mathfrak{C}_{qc}$  denote the class of all  $C^*$ -algebras  $A$  for which  $\text{Prim}(A)$  has the property  $(QC)$ .

**Corollary 10.23.** *Let  $A$  be a  $C^*$ -algebra. Then*

- (1) *If  $(J_\lambda)_{\lambda \in \Lambda}$  is an up-directed net of closed ideals of  $A$  such that  $\cup_{\lambda \in \Lambda} J_\lambda$  is dense in  $A$  and  $J_\lambda \in \mathfrak{C}_{qc}$  for all  $\lambda$ , then  $A \in \mathfrak{C}_{qc}$ ;*
- (2) *If  $A$  has an ideal  $J \in \mathfrak{C}_{qc}$  such that  $\text{Prim}(A/J)$  is Hausdorff, then  $A \in \mathfrak{C}_{qc}$ ;*
- (3) *If  $A \in \mathfrak{C}_{qc}$  then all closed ideals and all quotients of  $A$  belong to  $\mathfrak{C}_{qc}$ .*

*Proof.* (1) One has that

$$\text{Prim}(A) \cong \overline{\cup_{\lambda \in \Lambda} \text{Prim}(J_\lambda)},$$

if  $\text{Prim}(J_\lambda)$  is identified with the set of all  $I \in \text{Prim}(A)$  that do not contain  $J_\lambda$ . All  $\text{Prim}(J_\lambda)$  are open and have the property  $(QC)$ . Using Lemma 10.22(1) we get that  $\text{Prim}(A)$  has property  $(QC)$ . So  $A \in \mathfrak{C}_{qc}$ .

(2) In this case  $\text{Prim}(A) \cong \text{Prim}(J) \cup \text{Prim}(A/J)$  and it remains to apply Lemma 10.22(2).

(3) Follows from Lemma 10.22(3) because, for each closed ideal  $J$  of  $A$ , the space  $\text{Prim}(J)$  (respectively,  $\text{Prim}(A/J)$ ) is homeomorphic to an open (respectively, closed) subset of  $\text{Prim}(A)$ .  $\square$

**Corollary 10.24.** *All GCR  $C^*$ -algebras belong to  $\mathfrak{C}_{qc}$ .*

*Proof.* It is known [Dm, Proposition 4.5.3 and Theorem 4.5.5] that if  $A$  is a GCR algebra then there is an increasing transfinite chain  $(J_\alpha)_{\alpha \leq \delta}$  of closed ideals such that  $J_0 = 0$ ,  $J_\delta = A$  and all gap-quotients  $J_{\alpha+1}/J_\alpha$  have Hausdorff space of primitive ideals.

Assume, to the contrary, that there is a smallest ordinal  $\gamma$  for which  $J_\gamma \notin \mathfrak{C}_{qc}$ . If  $\gamma$  is limit then  $J_\gamma \in \mathfrak{C}_{qc}$  by Lemma 10.23(1); if not, then  $\gamma = \alpha + 1$  for some ordinal  $\alpha$ , whence  $J_\gamma \in \mathfrak{C}_{qc}$  by Lemma 10.23(2), a contradiction.  $\square$

At the moment we do not know an example of a  $C^*$ -algebra which does not belong to  $\mathfrak{C}_{qc}$ .

**Lemma 10.25.** *Let  $A$  be a  $C^*$ -algebra, and let  $M$  be a precompact subset of  $A$ . Then, for any  $t > 0$ , the set  $\{I \in \text{Prim}(A) : \|M/I\| \geq t\}$  is quasicompact.*

*Proof.* We may argue as in [Dm, Proposition 3.3.7], where the statement is proved for one-element sets, if we establish that for each closed ideal  $J$  of  $A$ , there is a primitive ideal  $I \supset J$  with  $\|M/I\| = \|M/J\|$ . To show the existence of  $I$ , let  $x_n \in M$  be such that

$$\|M/J\| = \lim_{n \rightarrow \infty} \|x_n/J\|.$$

By precompactness of  $M$  we may assume that  $x_n \rightarrow x \in A$ , and it remains to take  $I$  with  $\|x/I\| = \|x/J\|$  ([Dm, Lemma 3.3.6]).  $\square$

**Theorem 10.26.** *Let  $A$  be a  $C^*$ -algebra. If  $A \in \mathfrak{C}_{qc}$  then*

$$\rho(M) = \sup_{I \in \text{Prim}(A)} \rho(M/I)$$

*for each precompact subset  $M$  of  $A$ .*

*Proof.* For each  $n$ , let  $K_n = \{I \in \text{Prim}(A) : \|M^{2^n}/I\| \geq \rho(M)^{2^n}\}$ . Since

$$\|M^{2^n}\| \geq \rho(M)^{2^n},$$



all  $K_n$  are non-zero, and they are quasicompact by Lemma 10.25. Since the sequence  $\|M^{2^n}/I\|^{1/2^n}$  decreases, the inclusion

$$K_n \supset K_{n+1}$$

holds for each  $n$ . It follows that there is a primitive ideal  $I \in \cap_{n=1}^{\infty} K_n$ . Thus

$$\|M^{2^n}/I\|^{1/2^n} \geq \rho(M)$$

for all  $n$ . Taking the limit we get that  $\rho(M/I) \geq \rho(M)$ .  $\square$

**Corollary 10.27.** *Let  $A$  be a  $C^*$ -algebra, and let  $M$  be a precompact subset of  $A$ . If  $A \in \mathfrak{C}_{qc}$  and  $\rho(M/I) = r(M/I)$  for all  $I \in \text{Prim}(A)$  then*

$$\rho(M) = r(M).$$

*Proof.* By Theorem 10.26, we obtain that

$$\rho(M) = \sup_{I \in \text{Prim}(A)} \rho(M/I) = \sup_{I \in \text{Prim}(A)} r(M/I) \leq r(M).$$

$\square$

The following result establishes a  $C^*$ -version of the joint spectral radius formula for algebras satisfying the condition (10.14) with all their quotients.

**Proposition 10.28.** *Let  $A$  be a  $C^*$ -algebra. If, for any closed ideal  $K$  of  $A$ ,*

$$\rho(N) = \sup_{I \in \text{Prim}(A/K)} \rho(N/I)$$

*for every precompact subset  $N$  of  $A/K$  then*

$$\rho(M) = \max\{r(M), \rho(M/\mathcal{R}_{gcr}(A))\}$$

*for every precompact subset  $M$  of  $A$ .*

*Proof.* We show that

$$\rho(M) = \max\{r(M), \rho(M/J)\} \quad (10.20)$$

for each closed GCR ideal  $J$  of  $A$ .

Let us firstly prove (10.20) for the case that  $J$  is a CCR-ideal (assuming only that (10.14) is true for  $A$ ). By (10.14), it suffices to show that

$$\rho(M/I) \leq \max\{\rho(M/J), r(M)\}$$

for each  $I \in \text{Prim}(A)$ . Let  $\pi$  be an irreducible representation of  $A$  on a Hilbert space  $H$ , with  $\ker \pi = I$ . Then

$$\rho(M/I) = \rho(\pi(M)).$$

If  $\pi(J) = 0$  then  $\pi$  is a representation of  $A/J$  and the inequality

$$\rho(\pi(M)) \leq \rho(M/J)$$

is evident. Otherwise  $\pi(J) = \mathcal{K}(H)$  and

$$\rho_e(\pi(M)) = \rho(\pi(M)/\pi(J)) \leq \rho(M/J).$$

Applying the operator version of the joint spectral radius formula (2.10) to  $\pi(M)$  we obtain the needed inequality.

In general, we have an increasing transfinite chain  $(J_\alpha)_{\alpha \leq \delta}$  such that  $J_0 = 0$ ,  $J_\delta = J$  and all gap-quotients  $J_{\alpha+1}/J_\alpha$  are CCR-algebras.

Assume, to the contrary, that there is an ordinal  $\gamma$  which is the smallest one among ordinals  $\alpha$  for which (10.20) is not true with  $J = J_\alpha$ . If  $\gamma$  is not limit then  $\gamma = \alpha + 1$  for some  $\alpha$ . As  $M/J_\gamma \cong (M/J_\alpha)/(J_\gamma/J_\alpha)$  then

$$\rho(M/J_\alpha) = \max\{r(M/J_\alpha), \rho(M/J_\gamma)\}.$$

Since also  $\rho(M) = \max\{r(M), \rho(M/J_\alpha)\}$  then

$$\rho(M) = \max\{r(M), \rho(M/J_\gamma)\},$$

a contradiction.

Let now  $\gamma$  be a limit ordinal. By Lemma 9.14,

$$\begin{aligned} \max\{r(M), \rho(M/J_\gamma)\} &= \max\{r(M), \inf_{\alpha' < \gamma} \rho(M/J_{\alpha'})\} \\ &= \inf_{\alpha' < \gamma} \max\{r(M), \rho(M/J_{\alpha'})\} = \rho(M), \end{aligned}$$

a contradiction, so the equality holds for all ordinals.  $\square$

**Corollary 10.29.** *Let  $A$  be a  $C^*$ -algebra. If  $A \in \mathfrak{C}_{qc}$  then*

$$\rho(M) = \max\{r(M), \rho(M/\mathcal{R}_{gcr}(A))\}$$

*for every precompact subset  $M$  of  $A$ .*

*Proof.* Indeed, any quotient of a  $C^*$ -algebra from  $\mathfrak{C}_{qc}$  belongs to  $\mathfrak{C}_{qc}$  (see Corollary 10.23(3)). So the result follows from Theorems 10.28 and 10.26.  $\square$

**Corollary 10.30.** *Each GCR  $C^*$ -algebra is a Berger-Wang algebra.*

*Proof.* Follows from Corollary 10.29.  $\square$

**Corollary 10.31.** *Let  $A$  be a  $C^*$ -algebra. Then any precompact subset of  $\mathcal{R}_{gcr}(A)$  is a point of continuity of the joint spectral radius.*

*Proof.* Note that  $\mathcal{R}_{gcr}(A)$  is the largest GCR-ideal of  $A$  by Theorem 2.23. So the result follows from Corollary 10.30 and Theorems 9.11 and 10.17.  $\square$

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